STAT 511

FINAL EXAM SOLUTIONS

SPRING 2012

1.

a) $\exp(0.3747)$

b) $41.681 - 37.025$

c) $\hat{\phi} = 37.025/18$

d) $$t = \frac{0.3747}{\left(\sqrt{\frac{37.025}{18}}\right)(0.1752)}$$

e) 18

2. a) $-2(-700) + 2(25+2) = 1454.$

$-2(-675) + 2(25+6+5+4+3+2+1)$

$= 1442.$

b) THE UNSTRUCTURED VARIANCE OF MODEL 2 IS PREFERRED.

3.
$$\frac{\beta_1 \exp(x-\beta_2)}{1+\exp(x-\beta_2)} = \beta_1 \frac{1}{\exp(-x+\beta_2)+1}$$

$$\rightarrow \beta_1 \text{ AS } x \rightarrow \infty.$$

$\therefore$ 5 IS A REASONABLE STARTING VALUE FOR β_1.

WHEN $x = \beta_2$, $\dfrac{\beta_1 \exp(x-\beta_2)}{1+\exp(x-\beta_2)} = \beta_1/2$.

AT $y = 5/2 = 2.5$, $x \approx 8.5$, WHICH IS A REASONABLE STARTING VALUE FOR β_2.

4. a) $$\frac{(11-9)+(11-9-5+12)+(11-9+10+3)}{3}$$

$$= 8\tfrac{2}{3}$$

b) $$VAR(\text{lsmean}) = \left(\frac{1}{3}\right)^2\left(\frac{\sigma^2}{3}+\frac{\sigma^2}{1}+\frac{\sigma^2}{4}\right)$$

$$= \frac{19}{108}\sigma^2$$

$$\frac{\hat{\sigma}^2}{2} = (1.871)^2 \implies SE = \sqrt{\frac{19}{54}(1.871)^2}$$

5.

a) $$X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 1 & 7 & 1 & 0 \\ 1 & 9 & 3 & 0 \\ 1 & 11 & 5 & 0 \\ 1 & 13 & 7 & 0 \\ 1 & 15 & 9 & 1 \\ 1 & 17 & 11 & 3 \\ 1 & 19 & 13 & 5 \\ 1 & 21 & 15 & 7 \end{bmatrix}$$

5.

b) SLOPE FOR

$x < 6: \quad \hat{\beta}_2$

$x \in (6, 14): \quad \hat{\beta}_2 + \hat{\beta}_3$

$x > 14: \quad \hat{\beta}_2 + \hat{\beta}_3 + \hat{\beta}_4$

5. C) FALSE. PENALIZATION TENDS TO MAKE $\hat{\beta}_3$ AND $\hat{\beta}_4$ CLOSER TO ZERO, WHICH TENDS TO MAKE THE SLOPES MORE SIMILAR TO EACH OTHER THAN THE UNPENALIZED OLS ESTIMATES.

6.

a) $\underset{\sim}{y} = \begin{bmatrix} 51 \\ 54 \\ 48 \\ 52 \end{bmatrix}$ $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\underset{\sim}{\beta} = \begin{bmatrix} E(W_1) \\ E(W_2) \end{bmatrix}$

$\Sigma = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ $\underset{\sim}{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $\underset{\sim}{c}'(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}\underset{\sim}{y}$

7.

LET $\underset{\sim}{a}'\underset{\sim}{y}$ DENOTE ANY LINEAR UNBIASED ESTIMATOR OF μ. THEN

$$E(\underset{\sim}{a}'\underset{\sim}{y}) = \underset{\sim}{a}'\underset{\sim}{1}\mu = \mu \quad \forall\ \mu \in \mathbb{R}.$$

THUS, $\underset{\sim}{a}'\underset{\sim}{1} = 1 \iff \underset{\sim}{a}'\underset{\sim}{y}$ IS LINEAR UNBIASED ESTIMATOR OF μ.

NOTE THAT $\bar{y}_\cdot = \frac{1}{n}\underset{\sim}{1}'\underset{\sim}{y}$ IS $\therefore$ UNBIASED AND LINEAR.

ALSO,
$$\begin{aligned} \mathrm{VAR}(\underset{\sim}{a}'\underset{\sim}{y}) &= \mathrm{VAR}\left(\underset{\sim}{a}'\underset{\sim}{y} - \tfrac{1}{n}\underset{\sim}{1}'\underset{\sim}{y} + \tfrac{1}{n}\underset{\sim}{1}'\underset{\sim}{y}\right) \\ &= \mathrm{VAR}\left(\underset{\sim}{a}'\underset{\sim}{y} - \tfrac{1}{n}\underset{\sim}{1}'\underset{\sim}{y}\right) + \mathrm{VAR}\left(\tfrac{1}{n}\underset{\sim}{1}'\underset{\sim}{y}\right) \\ &\quad + 2\,\mathrm{COV}\left(\underset{\sim}{a}'\underset{\sim}{y} - \tfrac{1}{n}\underset{\sim}{1}'\underset{\sim}{y},\ \tfrac{1}{n}\underset{\sim}{1}'\underset{\sim}{y}\right). \end{aligned}$$

$$\mathrm{Cov}\left(\underline{a}'\underset{\sim}{y} - \tfrac{1}{n}\underline{1}'\underset{\sim}{y},\ \tfrac{1}{n}\underline{1}'\underset{\sim}{y}\right)$$

$$= \mathrm{Cov}\left[\left(\underline{a} - \tfrac{1}{n}\underline{1}\right)'\underset{\sim}{y},\ \tfrac{1}{n}\underline{1}'\underset{\sim}{y}\right]$$

$$= \left(\underline{a} - \tfrac{1}{n}\underline{1}\right)'\left[\sigma_1^2 I + \sigma_2^2 \underline{1}\,\underline{1}'\right]\left(\tfrac{1}{n}\underline{1}\right)$$

$$= \left(\underline{a} - \tfrac{1}{n}\underline{1}\right)'\left(\sigma_1^2 I\right)\left(\tfrac{1}{n}\underline{1}\right) + \left(\underline{a} - \tfrac{1}{n}\underline{1}\right)'\sigma_2^2\underline{1}\,\underline{1}'\left(\tfrac{1}{n}\underline{1}\right)$$

$$= \frac{\sigma_1^2}{n}\left(\underline{a} - \tfrac{1}{n}\underline{1}\right)'\underline{1} + \sigma_2^2\left(\underline{a} - \tfrac{1}{n}\underline{1}\right)'\underline{1}$$

$= 0$ BECAUSE $\left(\underline{a} - \tfrac{1}{n}\underline{1}\right)'\underline{1} = \underline{a}'\underline{1} - 1$

$$= 1 - 1$$

$$= 0.$$

$$\therefore \quad \mathrm{VAR}(\underset{\sim}{a}'\underset{\sim}{y}) = \mathrm{VAR}(\underset{\sim}{a}'\underset{\sim}{y} - \tfrac{1}{n}\underset{\sim}{1}'\underset{\sim}{y})$$

$$+ \mathrm{VAR}(\tfrac{1}{n}\underset{\sim}{1}'\underset{\sim}{y})$$

$$\geq \mathrm{VAR}(\tfrac{1}{n}\underset{\sim}{1}'\underset{\sim}{y})$$

WITH EQUALITY IFF $\underset{\sim}{a} = \frac{1}{n}\underset{\sim}{1}$.

$\therefore$ $\frac{1}{n}\underset{\sim}{1}'\underset{\sim}{y} = \bar{Y}.$ HAS THE SMALLEST VARIANCE AMONG ALL UNBIASED ESTIMATORS OF μ.