STAT 510

Final Exam Solutions

Spring 2017

1a) 12	3a) 10	5a) 10
1b) 8	3b) 10	5b) 10
2a) 8	4a) 12	
2b) 12	4b) 8	

1\. a) $\frac{Y_1+Y_2}{2}$ AND Y_3 ARE INDEPENDENT BLUES OF μ_1

$$VAR\left(\frac{Y_1+Y_2}{2}\right) = VAR\left(u_1 + \frac{e_1+e_2}{2}\right) = \sigma_u^2 + \sigma_e^2/2 = 4$$

$$VAR(Y_3) = VAR(u_2 + e_3) = \sigma_u^2 + \sigma_e^2 = 5$$

Y_4 IS OBVIOUSLY THE BLUE OF μ_2.

COMBINING ALL THIS TOGETHER, THE BLUE OF $\mu_1 - \mu_2$ IS

$$\frac{\frac{1}{4}\frac{Y_1+Y_2}{2} + \frac{1}{5}Y_3}{\frac{1}{4}+\frac{1}{5}} - Y_4$$

COMBINING INDEPENT BLUES BY INVERSE VARIANCE WEIGHTING IS THE OPTIMAL WAY TO LINEARLY COMBINE, BUT IT ISN'T ACTUALLY GUARANTEED TO RESULT IN THE BLUE COMPUTED FROM THE ORIGINAL RAW DATA. IT DOES IN THIS CASE, BUT SEE THE NEXT 3 PAGES FOR PROOF.

$$= \frac{5\frac{Y_1+Y_2}{2} + 4Y_3}{9} - Y_4 = \frac{5}{18}Y_1 + \frac{5}{18}Y_2 + \frac{4}{9}Y_3 - 1Y_4$$

$$\therefore a_1 = \frac{5}{18},\ a_2 = \frac{5}{18},\ a_3 = \frac{4}{9},\ a_4 = -1$$

WE CAN WRITE THE MODEL AS

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

$$\mathrm{VAR}(\underset{\sim}{y}) = ZGZ' + R$$

$$= \sigma_u^2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \sigma_e^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$[\mathrm{Var}(\underline{y})]^{-1} = \Sigma^{-1} = \begin{bmatrix} 5/16 & -3/16 & 0 & 0 \\ -3/16 & 5/16 & 0 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & & 1/5 \end{bmatrix}$$

$$X' \Sigma^{-1} = \begin{bmatrix} 1/8 & 1/8 & 1/5 & 0 \\ 0 & 0 & 0 & 1/5 \end{bmatrix}$$

$$X' \Sigma^{-1} X = \begin{bmatrix} 9/20 & 0 \\ 0 & 1/5 \end{bmatrix}$$

$$(X' \Sigma^{-1} X)^{-1} = \begin{bmatrix} 20/9 & 0 \\ 0 & 5 \end{bmatrix}$$

$$X' \Sigma^{-1} \underline{y} = \begin{bmatrix} \frac{1}{8} y_1 + \frac{1}{8} y_2 + \frac{1}{5} y_3 \\ \frac{1}{5} y_4 \end{bmatrix}$$

$$(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}\underset{\sim}{y} = \begin{bmatrix} 5/18\, y_1 + 5/18\, y_2 + 4/9\, y_3 \\ y_4 \end{bmatrix}$$

$$[1 \;,\; -1]\,(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}\underset{\sim}{y} =$$

$$\boxed{\frac{5}{18}y_1 + \frac{5}{18}y_2 + \frac{4}{9}y_3 - y_4}$$

1. b) $X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \underset{\sim}{a}_1' = [1 \ -1 \ 0 \ 0]$

$\underset{\sim}{a}_2' = [1 \ 1 \ -2 \ 0]$

$\Longrightarrow \underset{\sim}{a}_k' X = \underset{\sim}{0}' \quad \forall\, k = 1, 2$

AND $\underset{\sim}{a}_1, \underset{\sim}{a}_2$ NOT LINEARLY DEPENDENT BECAUSE $\underset{\sim}{a}_1 \neq c\underset{\sim}{a}_2 \ \forall\, c \in \mathbb{R}$

ERROR CONTRASTS ARE THUS

$Y_1 - Y_2$ AND $Y_1 + Y_2 - 2Y_3$.

INFINITELY MANY OTHER ANSWERS ARE ALSO ACCEPTABLE.

2\. a) $2 \times 44.95243 + 2 \times (8+2) \approx 109.9$ (AIC)

BECAUSE WE USE REML LIKELIHOOD

$2 \times 44.95243 + (8+2) \times \log(\underbrace{40-8})$ (BIC)

2\. b)

		FT1	FT2
LEFT FOOT	OM1	M	M+FT2
	OM2	M+OM2	M+OM2+FT2+OM2:FT2
RIGHT FOOT	OM1	M+footR	M+footR+FT2+footR:FT2
	OM2	M+footR+OM2+footR:OM2	M+footR+OM2+FT2+OM2:FT2 +footR:FT2+footR:OM2 +footR:OM2:FT2

2 b) (CONTINUED)

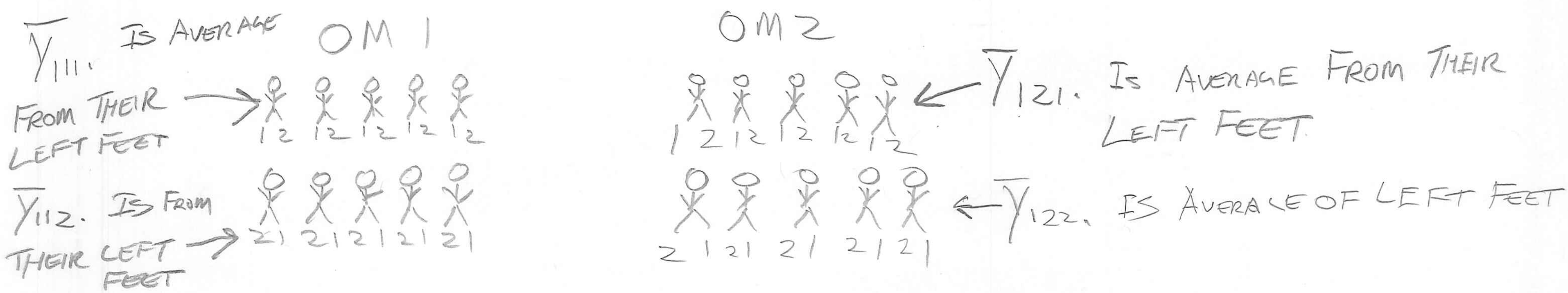
NOTE THAT

$$\mu + OM2 + FT2 + OM2{:}FT2 - (\mu + OM2) - (\mu + FT2) + \mu = OM2{:}FT2$$

THUS, $\bar{Y}_{122\cdot} - \bar{Y}_{121\cdot} - \bar{Y}_{112\cdot} + \bar{Y}_{111\cdot}$ IS THE BLUE.

THE VARIANCE OF THIS ESTIMATOR IS $\frac{4(\sigma_s^2+\sigma_e^2)}{5}$ BECAUSE

THESE ARE EACH AVERAGES OF 5 OBSERVATIONS FROM 5 DIFFERENT SUBJECTS, AND THE SUBJECTS IN EACH AVERAGE ARE INDEPENDENT OF THE SUBJECTS IN THE OTHER AVERAGES. THE SE IS $\sqrt{\frac{4}{5}\left(1.13231^2 + 0.3941764^2\right)}$.



3. a) $Z = -1.304$

p-VALUE $= 0.192$

THE TWO EXPECTED PROBABILITIES COULD BE THE SAME. WE SEE NO STATISTICALLY SIGNIFICANT DIFFERENCE BASED ON THESE DATA.

3. b) THE PLAYER WITH THE LOWEST PREDICTED SUCCESS PROBABILITY COMES FROM TEAM 12 WHICH WAS TAUGHT USING METHOD 2.
THE PREDICTED SUCCESS PROBABILITY IS

$$\left(1 + \exp\{-(0.1026 - 0.1779 - 0.23008532 - 1.002)\}\right)^{-1}$$

4. a) $X = \underset{2\times 2}{I} \otimes \underset{8\times 1}{\mathbf{1}}$

INSTRUCTION SET IS CLEARLY THE FACTOR OF INTEREST. WE WANT TO TEST FOR DIFFERENCES BETWEEN INSTRUCTION SET 1 AND 2. THUS, WE NEED FIXED EFFECTS FOR INSTRUCTION SETS.

$Z = \left[\underset{8\times 8}{I} \otimes \underset{2\times 1}{\mathbf{1}},\ A\right]$, WHERE

WE WANT RANDOM EFFECTS FOR PARTICIPANTS AND IMAGES BECAUSE WE ARE NOT INTERESTED IN JUST THESE 8 PARTICIPANTS OR JUST THESE 8 IMAGES. WE WANT TO GENERALIZE OUR RESULTS TO ALL PARTICIPANTS AND IMAGES.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \underset{14\times 1}{\mathbf{0}} & \multicolumn{7}{l}{\underset{7\times 7}{I} \otimes \underset{2\times 1}{\mathbf{1}}} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

MANY FORGOT TO SPECIFY INDEPENDENCE BETWEEN $\mathbf{u}$ AND $\mathbf{e}$.

4. b) $\mathbf{u} = \begin{bmatrix} \underset{8\times 1}{\mathbf{u}_1} \\ \underset{8\times 1}{\mathbf{u}_2} \end{bmatrix}$ $\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{e} \end{bmatrix} \sim N\left(\mathbf{0}, \begin{bmatrix} \sigma_P^2 \underset{8\times 8}{I} & 0 & 0 \\ 0 & \sigma_I^2 \underset{8\times 8}{I} & 0 \\ 0 & 0 & \sigma_e^2 \underset{16\times 16}{I} \end{bmatrix}\right.$

5. a) $F = \dfrac{(3 + 17.4 + 180.2 + 4.4 + 3.3)/5}{222.8/50}$

5. b) FROM PAST HOMEWORK ASSIGNMENT, WE KNOW THE BLUE IS $\bar{Y}_{11\cdot} - \bar{Y}_{15\cdot} = \frac{1}{6}\sum_{k=1}^{6}(Y_{11k} - Y_{15k})$. THUS,

$$\begin{aligned} VAR(\bar{Y}_{11\cdot} - \bar{Y}_{15\cdot}) &= \frac{1}{6} VAR(Y_{111} - Y_{151}) \\ &= \frac{1}{6}(\sigma^2 + \sigma^2 - 2\sigma^2\rho^4) \\ &= \frac{\sigma^2}{3}(1-\rho^4). \end{aligned}$$