Exam 1 Solutions

Spring 2015

1a) 7

1b) 7

2a) 6

2b) 5

2c) 8

2d) 8

2e) 7

3a) 8

3b) 5

3c) 8

4a) 7

4b) 6

4c) 6

4d) 7

4e) 5

1a) $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ OR $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$ FOR EXAMPLE.

2a) $\hat{\mu}_{11} = \frac{3+5}{2} = 4$ $\quad \hat{\mu}_{12} = \frac{7+8+9}{3} = 8$

$\hat{\mu}_{21} = \frac{13}{1} = 13$ $\quad \hat{\mu}_{22} = \frac{2+4}{2} = 3$

$$\hat{\sigma}^2 = \left[(3-4)^2 + (5-4)^2 + (7-8)^2 + (8-8)^2 + (9-8)^2 + (13-13)^2 + (2-3)^2 + (4-3)^2\right] / (8-4)$$

$$= 6/4 = 1.5$$

2b) A1: $\frac{\hat{\mu}_{11} + \hat{\mu}_{12}}{2} = \frac{4+8}{2} = 6$

A2: $\frac{\hat{\mu}_{21} + \hat{\mu}_{22}}{2} = \frac{13+3}{2} = 8$

2c) $(C\hat{\beta})'(C(X'X)^{-1}C')^{-1}C\hat{\beta} = \text{TYPE III SS}$, WHERE

$C = [1, 1, -1, -1]$, $\hat{\beta} = (\hat{\mu}_{11}, \hat{\mu}_{12}, \hat{\mu}_{21}, \hat{\mu}_{22})'$

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad X'X = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad (X'X)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$C(X'X)^{-1}C' = 1/2 + 1/3 + 1 + 1/2 = 7/3 \quad (C(X'X)^{-1}C')^{-1} = 3/7$

$C\hat{\beta} = -4 \qquad \text{TYPE III SS} = (-4)^2\, 3/7 = \frac{48}{7} = 6\frac{6}{7}.$

2d) MULTIPLYING THE MODEL MATRIX TIMES [Intercept, A2, A1:B2, A2:B2]'

GIVES $\hat{E}(y) = \begin{bmatrix} \text{Intercept} \\ \text{Intercept} \\ \text{Intercept} + \text{A1:B2} \\ \text{Intercept} + \text{A1:B2} \\ \text{Intercept} + \text{A1:B2} \\ \text{Intercept} + \text{A2} \\ \text{Intercept} + \text{A2} + \text{A2:B2} \\ \text{Intercept} + \text{A2} + \text{A2:B2} \end{bmatrix}$

THUS,
$$\left.\begin{aligned} \text{Intercept} &= \hat{\mu}_{11} \\ \text{Intercept} + \text{A1:B2} &= \hat{\mu}_{12} \\ \text{Intercept} + \text{A2} &= \hat{\mu}_{21} \\ \text{Intercept} + \text{A2} + \text{A2:B2} &= \hat{\mu}_{22} \end{aligned}\right\} \Longleftrightarrow \begin{aligned} \text{Intercept} &= \hat{\mu}_{11} = 4 \\ \text{A1:B2} &= \hat{\mu}_{12} - \hat{\mu}_{11} = 8-4=4 \\ \text{A2} &= \hat{\mu}_{21} - \hat{\mu}_{11} = 13-4=9 \\ \text{A2:B2} &= \hat{\mu}_{22} - \hat{\mu}_{21} = 3-13=-10 \end{aligned}$$

SEs ARE $\underset{(\text{Intercept})}{\sqrt{1.5\,\frac{1}{2}}}$, $\underset{(\text{A1:B2})}{\sqrt{1.5\left(\frac{1}{3}+\frac{1}{2}\right)}}$, $\underset{(\text{A2})}{\sqrt{1.5\left(\frac{1}{1}+\frac{1}{2}\right)}}$, $\underset{(\text{A2:B2})}{\sqrt{1.5\left(\frac{1}{2}+\frac{1}{1}\right)}}$

2e) EXAMINING THE ROWS OF THE MODEL MATRIX REVEALS THAT THIS MODEL RESTRICTS μ_{11} TO BE THE SAME AS μ_{21}. μ_{12} AND μ_{22} ARE UNRESTRICTED. THUS, THIS IS A REDUCED MODEL FOR TESTING FOR A SIMPLE EFFECT OF FACTOR A WHEN FACTOR B IS FIXED AT LEVEL B1. ($H_0: \mu_{11} = \mu_{21}$).

3 a) LET $X_1 = \underset{\sim}{1}_{40\times 1}$ AND $X_2 = \underset{4\times 4}{I} \otimes \underset{10\times 1}{\underset{\sim}{1}}$. THEN

$P_{X_1} = P_1 = \frac{1}{40} \underset{\sim}{1}\underset{\sim}{1}'$ AND $P_{X_2} = P_2 = \underset{4\times 4}{I} \otimes \frac{1}{10} \underset{\sim}{1}\underset{\sim}{1}'$

RANK $(X_1) = 1$ AND RANK $(X_2) = 4$. LET $\underset{\sim}{\beta} = (\mu_1, \mu_2, \mu_3, \mu_4)'$.

LET $\underset{\sim}{y} = (y_{11}, y_{12}, \cdots y_{1,10}, y_{21}, y_{22}, \cdots y_{2,10}, \cdots y_{4,10})'$.

$$F = \frac{\underset{\sim}{y}'(P_2 - P_1)\underset{\sim}{y} / (4-1)}{\underset{\sim}{y}'(I - P_2)\underset{\sim}{y} / (40-4)} = \frac{\|(P_2 - P_1)\underset{\sim}{y}\|^2 / 3}{\|(I - P_2)\underset{\sim}{y}\|^2 / 36}$$

$$= \frac{\|P_2\underset{\sim}{y} - P_1\underset{\sim}{y}\|^2 / 3}{\|\underset{\sim}{y} - P_2\underset{\sim}{y}\|^2 / 36} = \frac{\frac{1}{3}\sum_{i=1}^{4}\sum_{j=1}^{10}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2}{\frac{1}{36}\sum_{i=1}^{4}\sum_{j=1}^{10}(Y_{ij} - \bar{Y}_{i\cdot})^2}$$

$$= \frac{120\sum_{i=1}^{4}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2}{\sum_{i=1}^{4}\sum_{j=1}^{10}(Y_{ij} - \bar{Y}_{i\cdot})^2}$$

3 b) 3 AND 36

3c) $\frac{\underset{\sim}{\beta}' X_2' (P_2 - P_1) X_2 \underset{\sim}{\beta}}{2\sigma^2} = \frac{\| (P_2 - P_1) X_2 \underset{\sim}{\beta} \|^2}{2\sigma^2} = \frac{\| P_2 X_2 \underset{\sim}{\beta} - P_1 X_2 \underset{\sim}{\beta} \|^2}{2\sigma^2}$

$$= \frac{10 \sum_{i=1}^{4} (\mu_i - \bar{\mu}_\cdot)^2}{2\sigma^2} = \frac{5 \sum_{i=1}^{4} (\mu_i - \bar{\mu}_\cdot)^2}{\sigma^2}$$

$$= 2 \left[(9 - 6.5)^2 + (7 - 6.5)^2 + 2(5 - 6.5)^2 \right]$$

$$= 2 \left[2.5^2 + .5^2 + 2 \times 1.5^2 \right]$$

$$= 2 \left[6.25 + .25 + 2 \times 2.25 \right]$$

$$= 22$$

3 c) ALTERNATIVE SOLUTION:

$$NCP = \frac{(C\underset{\sim}{\beta})'(C(X'X)^{-1}C')^{-1}C\underset{\sim}{\beta}}{2\sigma^2}, \quad \text{WHERE } X = X_2 \text{ AND}$$

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix}.$$

$$C(X'X)^{-1}C' = \frac{1}{10}CC' = \frac{1}{10}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$(C(X'X)^{-1}C')^{-1} = 10\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/12 \end{bmatrix}$$

$$C\underset{\sim}{\beta} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 9-7 \\ 9+7-2\times5 \\ 9+7+5-3\times5 \end{bmatrix}$$

$$(C\underset{\sim}{\beta})'(C(X'X)^{-1}C')^{-1}C\underset{\sim}{\beta} = 10\left[\frac{2^2}{2} + \frac{6^2}{6} + \frac{6^2}{12}\right] = 110$$

$$NCP = \frac{110}{2(2.5)} = 22.$$

4 a) $F = \dfrac{(62.5 + 64.3 + 56.0)/3}{27.2}$

b) F WITH 3 AND 6 DF.

c) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 20 \\ 1 & 20 \\ 1 & 40 \\ 1 & 40 \\ 1 & 50 \\ 1 & 50 \\ 1 & 50 \\ 1 & 50 \\ 1 & 50 \\ 1 & 50 \end{bmatrix}$

d) $F = \dfrac{(305.4 - 163.0)/4}{27.2}$

e) 4 AND 6