STAT 510

Exam 1 Solutions

Spring 2018

Points Possible:

1.	16
2. a)	12
2. b)	12
3.	14
4. a)	14
4. b)	32
Total	100

1. $X\underline{\beta} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \beta_1 + \beta_2 \\ \beta_1 + \beta_2 \\ \beta_1 + \beta_2 \\ \beta_1 + \beta_2 \\ \beta_1 - \beta_2 \\ \beta_1 - \beta_2 \\ \beta_1 - \beta_2 \\ \beta_1 - \beta_2 \end{bmatrix}$

CLEARLY, $\mu_1 = \beta_1 + \beta_2$ AND $\mu_2 = \beta_1 - \beta_2$.

THUS, $\mu_1 - \mu_2 = \beta_1 + \beta_2 - (\beta_1 - \beta_2)$

$$= 2\beta_2$$

$$= [0, 2]\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$= \underline{c}'\underline{\beta}, \text{ WITH } \underline{c}' = [0, 2].$$

$$\hat{\underline{\beta}} = (X'X)^{-1}X'\underline{y} = \frac{1}{8} I X'\underline{y} = \begin{bmatrix} 52.4/8 \\ 7.6/8 \end{bmatrix}$$

$$\underline{c}'\hat{\underline{\beta}} = 7.6/4 = 1.9$$

1. (CONTINUED)

$$SE(\underline{c}'\hat{\beta}) = \sqrt{\hat{\sigma}^2 \underline{c}'(X'X)^{-1}\underline{c}}$$

$$= \sqrt{\frac{2.52}{8-2}[0,2]\,\tfrac{1}{8}I\begin{bmatrix}0\\2\end{bmatrix}}$$

$$= \sqrt{0.42/2}$$

$$= \sqrt{0.21}$$

$$1.9 \pm 2.45\sqrt{0.21}$$

2. a)

μ_{11}	μ_{12}
μ_{21}	μ_{22}

$\Longleftrightarrow$

μ	$\mu + B2$
$\mu + A2$	$\mu + A2 + B2 + A2{:}B2$

$$\mu = \mu_{11}$$

$$A2 = \mu + A2 - \mu = \mu_{21} - \mu_{11}$$

$$B2 = \mu + B2 - \mu = \mu_{12} - \mu_{11}$$

$$A2{:}B2 = \mu + A2 + B2 + A2{:}B2 - (\mu + A2) - (\mu + B2) + \mu$$

$$= \mu_{22} - \mu_{21} - \mu_{12} + \mu_{11}$$

Thus, coef(o) yields

$$\hat{\mu} = \hat{\mu}_{11} = 5.9$$

$$\hat{A2} = \hat{\mu}_{21} - \hat{\mu}_{11} = 2.2 - 5.9 = -3.7$$

$$\hat{B2} = \hat{\mu}_{12} - \hat{\mu}_{11} = 3.4 - 5.9 = -2.5$$

$$\widehat{A2{:}B2} = 2.5 - 2.2 - 3.4 + 5.9 = 2.8$$

2 b) $\widehat{Cov}(\hat{B2}, \widehat{A2{:}B2})$

$$= \widehat{Cov}(\bar{Y}_{12\cdot} - \bar{Y}_{11\cdot}, \ \bar{Y}_{22\cdot} - \bar{Y}_{21\cdot} - \bar{Y}_{12\cdot} + \bar{Y}_{11\cdot})$$

$$= \widehat{Cov}\left[(-1, 1, 0, 0)\,\hat{\underline{\mu}}, \ (1, -1, -1, 1)\,\hat{\underline{\mu}}\right]$$

$$= [-1, 1, 0, 0] \ VAR(\hat{\underline{\mu}}) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

2b) (CONTINUED)

$$= [-1, 1, 0, 0] \frac{\hat{\sigma}^2}{5} I \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \frac{\hat{\sigma}^2}{5} [-1, 1, 0, 0] \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \frac{-2\hat{\sigma}^2}{5} = \frac{-2}{5} \frac{15}{20-4} = -3/8$$

3. $\mathrm{VAR}(\bar{Y}_\cdot) = \frac{1}{4}\mathrm{VAR}(Y_1 + Y_2)$

$$= \frac{1}{4}\mathrm{VAR}\left([1,1]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\right)$$

$$= \frac{1}{4}[1,1]\begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4}(1 + 1/4) = \frac{1.25}{4} = 5/16$$

OR $\mathrm{VAR}(\bar{Y}_\cdot) = \frac{1}{4}\mathrm{VAR}(Y_1 + Y_2) = \frac{1}{4}[\mathrm{VAR}(Y_1) + \mathrm{VAR}(Y_2)]$

$$= \frac{1}{4}(1 + 1/4) = 5/16$$

Y_1 IS A LINEAR ESTIMATOR BECAUSE IT IS OF THE FORM $\underline{a}'\underline{y}$ (WHERE $\underline{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ IS FIXED AND $\underline{y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$). Y_1 IS AN UNBIASED ESTIMATOR OF μ BECAUSE $E(Y_1) = \mu$.

$$\mathrm{VAR}(Y_1) = 1/4 = 4/16 < 5/16 = \mathrm{VAR}(\bar{Y}).$$

THUS, Y_1 IS A LINEAR UNBIASED ESTIMATOR OF μ WITH LOWER VARIANCE THAN $\bar{Y}_\cdot$. SO, $\bar{Y}_\cdot$ CANNOT BE THE BLUE OF μ.

4 a) THE EQUATION OF THE PIECEWISE LINEAR FUNCTION ON $x \in (-\infty, 30]$ IS $\beta_0 + \beta_1 x$.

ON $x \in [30, \infty)$, WE HAVE A LINE WITH SLOPE β_2 THAT PASSES THROUGH THE POINT $(30, \beta_0 + 30\beta_1)$. THUS, THE INTERCEPT ON $x \in [30, \infty)$ IS $\beta_0 + 30\beta_1 - 30\beta_2$, WHICH MAKES THE EQUATION OF THE LINE

$$\beta_0 + 30\beta_1 - 30\beta_2 + \beta_2 x = \beta_0 + 30\beta_1 + \beta_2(x-30).$$

THUS,

$$\underline{x}_1 = \left[\begin{array}{c} 10\\10\\10\\20\\20\\20\\30\\30\\30\\30\\30\\30\\30\\30\\30 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} \text{MULTIPLE ON } \beta_1 \text{ FOR } x \le 30 \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} \text{MULTIPLE ON } \beta_1 \text{ FOR } x > 30 \end{array}$$

$$\underline{x}_2 = \left[\begin{array}{c} 0\\0\\0\\0\\0\\0\\0\\0\\0\\10\\10\\10\\20\\20\\20 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} \text{MULTIPLE ON } \beta_2 \text{ FOR } x \le 30 \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \text{MULTIPLE ON } \beta_2 \text{ FOR } x = 40 \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \text{MULTIPLE ON } \beta_2 \text{ FOR } x = 50 \end{array}$$

4b)

Source	SS	DF
Linear	$95.1 - 51.9 = 43.2$	$2-1=1$
Piecewise Linear	?	$3-2=1$
Cell Means	?	$5-3=2$
Error	7.5	$15-5=10$
C. Total	95.1	$15-1=14$

THE ABOVE ENTRIES ARE STRAIGHTFORWARD.
HOW DO WE GET THE TWO MISSING ENTRIES?
THE CELL MEANS LINE WILL TEST FOR LACK OF FIT OF THE PIECEWISE LINEAR MODEL.
WE KNOW THAT SUM OF SQUARES IS EQUAL TO
$(C\hat{\beta})'[C(X'X)^{-1}C']^{-1}C\hat{\beta}$ FOR APPROPRIATE C.

IF PIECEWISE LINEAR MODEL HOLDS,

$$\mu_{20} - \mu_{10} = \mu_{30} - \mu_{20} \quad \text{AND} \quad \mu_{30} - \mu_{40} = \mu_{40} - \mu_{50}$$

i.e., $-\mu_{10} + 2\mu_{20} - \mu_{30} = 0$ AND $\mu_{30} - 2\mu_{40} + \mu_{50} = 0$

THUS, $C = \begin{bmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$. $C\hat{\beta} = \begin{bmatrix} -12 + 2 \times 16 + 19 \\ 19 - 2 \times 18 + 17 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

4b) (CONTINUED)

$$X'X = 3I_{5\times 5}, \quad (X'X)^{-1} = \frac{1}{3}I$$

$$C(X'X)^{-1}C' = \frac{1}{3}CC' = \frac{1}{3}\begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix}$$

$$\left[C(X'X)^{-1}C'\right]^{-1} = \frac{3}{35}\begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}$$

$$(C\hat{\underline{\beta}})'\left[C(X'X)^{-1}C'\right]^{-1}C\hat{\underline{\beta}} = 18/35$$

THUS, THE SUM OF SQUARES FOR CELL MEANS IS 18/35.

THE SUM OF SQUARES FOR PIECEWISE LINEAR IS

$$95.1 - (43.2 + 18/35 + 7.5) \quad \text{OR}$$

$$44.4 - 18/35.$$