# EXAM 1 SOLUTIONS

# SPRING 2020

## POINT VALUES

1. 10

2. a) 6
   b) 8
   c) 8
   d) 10
   e) 8
   f) 6

3. a) 7
   b) 7
   c) 7
   d) 7

4. a) 8
   b) 8

1. $\underset{\sim}{x} \in \mathcal{C}(X) \implies X\underset{\sim}{v} = \underset{\sim}{x}$ FOR SOME $\underset{\sim}{v}$

$\implies WA\underset{\sim}{v} = \underset{\sim}{x}$ (BECAUSE $X = WA$)

$\implies W\underset{\sim}{u} = \underset{\sim}{x}$ FOR SOME $\underset{\sim}{u}$ ($\underset{\sim}{u} = A\underset{\sim}{v}$, e.g.)

$\implies \underset{\sim}{x} \in \mathcal{C}(W)$

$\therefore \mathcal{C}(X) \subseteq \mathcal{C}(W)$ [1]

$\underset{\sim}{w} \in \mathcal{C}(W) \implies W\underset{\sim}{u} = \underset{\sim}{w}$ FOR SOME $\underset{\sim}{u}$

$\implies XB\underset{\sim}{u} = \underset{\sim}{w}$ (BECAUSE $W = XB$)

$\implies X\underset{\sim}{v} = \underset{\sim}{w}$ FOR SOME $\underset{\sim}{v}$ ($\underset{\sim}{v} = B\underset{\sim}{u}$, e.g.)

$\implies \underset{\sim}{w} \in \mathcal{C}(X)$.

$\therefore \mathcal{C}(W) \subseteq \mathcal{C}(X)$. [2]

[1] AND [2] TOGETHER IMPLY $\mathcal{C}(X) = \mathcal{C}(W)$.

2 a) $X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 30 \\ 1 & 1 & 0 & 60 \\ 1 & 1 & 0 & 90 \\ 1 & 1 & 0 & 120 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 30 \\ 1 & 0 & 1 & 60 \\ 1 & 0 & 1 & 90 \\ 1 & 0 & 1 & 120 \end{bmatrix}$

b) $E(y_{11}) = \mu + \alpha_1$.

THUS, $\mu + \alpha_1$ IS ESTIMABLE.

$E(y_{12}) = \mu + \alpha_1 + 30\beta$

THUS, $\mu + \alpha_1 + 30\beta$ IS ESTIMABLE.

THUS, $\forall\ x \in \mathbb{R}$,

$$\frac{30-x}{30}(\mu + \alpha_1) + \frac{x}{30}(\mu + \alpha_1 + 30\beta) = \mu + \alpha_1 + \beta x$$

IS ESTIMABLE.

2c) $X \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/30 \end{bmatrix} = W$

THERE ARE INFINITELY MANY CORRECT ANSWERS

$W \begin{bmatrix} 1 & 1/2 & 1/2 & 60 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 30 \end{bmatrix} = X$

BY PROBLEM 1, $\mathcal{C}(X) = \mathcal{C}(W)$.

2d) LET $B = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/30 \end{bmatrix}$ SO $XB = W$

BY c), WE KNOW $P_X \underset{\sim}{y} = P_W \underset{\sim}{y}$.

THUS, $X\hat{\underset{\sim}{\beta}} = X(X'X)^{-}X'\underset{\sim}{y} = P_X \underset{\sim}{y} = P_W \underset{\sim}{y}$

$= W(W'W)^{-}W'\underset{\sim}{y}$

$= XB(W'W)^{-}W'\underset{\sim}{y}$

2d) (CONTINUED)

IT FOLLOWS THAT, FOR ANY ESTIMABLE $\underset{\sim}{c}'\underline{\beta}$

THE OLS ESTIMATOR

$$\underline{c}'\hat{\underline{\beta}} = \underset{\sim}{a}'X\hat{\underline{\beta}} = \underset{\sim}{a}'XB(W'W)^{-}W'\underline{y}$$
$$= \underset{\sim}{c}'B(W'W)^{-}W'\underline{y}$$

ALTERNATIVELY, THIS CAN BE SEEN BY NOTING

THAT $B(W'W)^{-}W'y$ IS A SOLUTION TO THE

NORMAL EQUATIONS:

$$X'XB(W'W)^{-}W'\underline{y} = X'W(W'W)^{-}W'\underline{y}$$
$$= X'P_W\underline{y}$$
$$= X'P_X\underline{y}$$
$$= X'\underline{y}.$$

THUS, $\forall$ ESTIMABLE $\underline{c}'\underline{\beta}$, THE OLS ESTIMATOR IS

$\underline{c}'B(W'W)^{-}W'\underline{y}$.

$$(W'W)^{-} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 20 \end{bmatrix}^{-1} = \begin{bmatrix} 1/10 & 0 & 0 \\ 0 & 1/10 & 0 \\ 0 & 0 & 1/20 \end{bmatrix}$$

2d) (CONTINUED)

$$W'y = \begin{bmatrix} 81 \\ 11 \\ 40 \end{bmatrix} \Rightarrow (W'W)^{-}W'y = \begin{bmatrix} 8.1 \\ 1.1 \\ 2.0 \end{bmatrix}$$

$$\Rightarrow B(W'W)^{-}W'y = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/30 \end{bmatrix} \begin{bmatrix} 8.1 \\ 1.1 \\ 2.0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 2.2 \\ 1/15 \end{bmatrix}$$

THUS, $\widehat{\mu + \alpha_1 + \beta x} = 3 + 0 + \frac{1}{15}x$

$$= 3 + \frac{1}{15}x.$$

2 e) $$\frac{(4.8-2.4)/[(10-3)-(10-6)]}{2.4/(10-6)} = 4/3$$

f) Numerator $= 3$

Denominator $= 4$

3a) $$\underset{\sim}{z}' M(M'M)^{-} M' \underline{z} = \underline{z}\, P_M \underline{z}$$

b) $Var(\underset{\sim}{z}) = I$, which is positive definite

because $\underset{\sim}{x}' I \underset{\sim}{x} = \underset{\sim}{x}'\underset{\sim}{x} = \sum_{i=1}^{m} x_i^2 > 0 \;\; \forall\, \underset{\sim}{x} \neq \underline{0}$.

$P_M = P_M'$ and $Rank(P_M) = Rank(M) = r$.

Thus, $P_M$ is symmetric matrix of rank $r$.

$P_M I P_M I = P_M I$ and $E(\underset{\sim}{z}) = \underline{0}$. Thus,

$$\underset{\sim}{z}' P_M \underset{\sim}{z} \sim \chi_r^2 \left(\underline{0}' P_M \underline{0}/2\right) = \chi_r^2 .$$

c) $$\frac{a/r}{(b-a)/(m-r)}$$

3 d) NOTE $b - a = \underset{\sim}{z}'\underset{\sim}{z} - \underset{\sim}{z}' P_M \underset{\sim}{z}$

$= \underset{\sim}{z}'(I - P_M)\underset{\sim}{z}$

$I - P_M = (I - P_M)'$

$\text{RANK}(I - P_M) = m - r$

$(I - P_M) I (I - P_M) I = (I - P_M) I$

THUS, $\underset{\sim}{z}'(I - P_M)\underset{\sim}{z} \sim \chi^2_{m-r}$

$P_M I (I - P_M) = P_M - P_M P_M = P_M - P_M = 0.$

THUS $a \perp b - a$

IT FOLLOWS THAT

$$\frac{a/r}{(b-a)/(m-r)} \sim F_{r,\, m-r}.$$

4a)

DRUG

| | | 1 | 2 | |
|---|---|---|---|---|
| DIET | 1 | $\mu$ | $\mu + drug2$ | $\mu + drug2/2$ |
| | 2 | $\mu + diet2$ | $\mu + diet2 + drug2 + diet2{:}drug2$ | $\mu + diet2 + drug2/2 + diet2{:}drug2/2$ |

THEREFORE, LSMEANS ARE

$$\text{DIET 1: } 4.9 + \frac{-4.4}{2} = 2.7$$

$$\text{DIET 2: } 4.9 - 5.1 + \frac{-4.4}{2} + \frac{4.3}{2} = -.25$$

4b) THE INTERACTION IS

$$\mu_{11} - \mu_{12} - \mu_{21} + \mu_{22}$$

IN R's PARAMETERIZATION, THAT'S

$$\mu - (\mu + drug2) - (\mu + diet2) + (\mu + drug2 + diet2 + diet2{:}drug2)$$

$$= diet2{:}drug2$$

THE OLS ESTIMATOR IS 4.3 FROM THE R CODE. BECAUSE THE OLS ESTIMATOR OF $\mu_{11} - \mu_{12} - \mu_{21} + \mu_{22}$ IS $\bar{Y}_{11.} - \bar{Y}_{12.} - \bar{Y}_{21.} + \bar{Y}_{22.}$, WE KNOW SE IS

$$\sqrt{\widehat{VAR}(\bar{Y}_{11.} - \bar{Y}_{12.} - \bar{Y}_{21.} + \bar{Y}_{22.})}$$

$$= \sqrt{\hat{\sigma}^2\left(\frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10}\right)}$$

$$= \sqrt{\frac{454}{36}\,\frac{4}{10}} = \sqrt{\frac{45.4}{9}}$$

4 c) (CONTINUED)

$$t = \frac{4.3}{\sqrt{45.4/9}} = \frac{12.9}{\sqrt{45.4}}$$