EXAM 2 SOLUTIONS SPRING 2014

POINTS WERE ASSIGNED AS FOLLOWS:

1a) 6
b) 6
c) 6

2. 13

3. a) 17
b) 4

4a) 7
b) 7

5a) 5
b) 8
c) 5
d) 8
e) 8

1. This question concerns the Aitken model.

We can consider the transformed model

$$\underbrace{V^{-1/2}\underset{\sim}{y}}_{\underset{\sim}{z}} = \underbrace{V^{-1/2}X}_{W}\underset{\sim}{\beta} + \underbrace{V^{-1/2}\underset{\sim}{\varepsilon}}_{\underset{\sim}{\delta}}$$

$$\underset{\sim}{z} = W\underset{\sim}{\beta} + \underset{\sim}{\delta}, \quad \text{where } \underset{\sim}{\delta} \sim N(\underset{\sim}{0}, \sigma^2 I).$$

This transformed model follows the normal-error Gauss-Markov model, for which we have many results.

a) $C\hat{\underset{\sim}{\beta}}_V = C(W'W)^- W'\underset{\sim}{z} = C(X'V^{-1}X)^- X'V^{-1}\underset{\sim}{y}$

b) $C\hat{\underset{\sim}{\beta}}_V \sim N(C\underset{\sim}{\beta}, \sigma^2 C(W'W)^- C')$

$\sim N(C\underset{\sim}{\beta}, \sigma^2 C(X'V^{-1}X)^- C')$

1c) $\underset{\sim}{c}'\hat{\underset{\sim}{\beta}}_V \pm t_{0.975,\, n-r}\sqrt{\hat{\sigma}_V^2\, \underset{\sim}{c}'(X'V^{-1}X)^{-}\underset{\sim}{c}}$,

where $\hat{\sigma}_V^2 = \dfrac{(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}_V)'V^{-1}(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}_V)}{n-r}$, $r = \text{RANK}(X)$.

See Aitken Model slides 10 & 11 for the derivation of $\hat{\sigma}_V^2$.

2. $\overline{y}_{\cdot jk} - \overline{y}_{\cdot j\cdot} - \overline{y}_{\cdot\cdot k} + \overline{y}_{\cdot\cdot\cdot}$

$= \overline{\mu}_{\cdot j} - \overline{\mu}_{\cdot j} - \overline{\mu}_{\cdot\cdot} + \overline{\mu}_{\cdot\cdot}$

$+ b_k - \overline{b}_{\cdot} - b_k + \overline{b}_{\cdot}$

$+ \overline{w}_{\cdot k} - \overline{w}_{\cdot\cdot} - \overline{w}_{\cdot k} + \overline{w}_{\cdot\cdot}$

$+ \overline{e}_{\cdot jk} - \overline{e}_{\cdot j\cdot} - \overline{e}_{\cdot\cdot k} + \overline{e}_{\cdot\cdot\cdot}$

$= \overline{e}_{\cdot jk} - \overline{e}_{\cdot j\cdot} - \overline{e}_{\cdot\cdot k} + \overline{e}_{\cdot\cdot\cdot}$

Now, for any given value of k, let

$a_j = \overline{e}_{\cdot jk} - \overline{e}_{\cdot j\cdot}$ and note that

a_1, a_2, a_3, a_4 are independent and average to

$$\bar{a}_{\bullet} = \frac{1}{4}(a_1 + a_2 + a_3 + a_4) = \bar{e}_{\bullet\bullet k} - \bar{e}_{\bullet\bullet\bullet}$$

Thus, $\sum_{j=1}^{4} (a_j - \bar{a}_{\bullet})^2 = \sum_{j=1}^{4} (\bar{e}_{\bullet jk} - \bar{e}_{\bullet j\bullet} - \bar{e}_{\bullet\bullet k} + \bar{e}_{\bullet\bullet\bullet})^2$

has expected value $(4-1)\,\sigma_a^2$ (by Slide 8 of slideset 11)

where $\sigma_a^2 = \text{Var}(a_1) = \text{Var}(\bar{e}_{\bullet 1k} - \bar{e}_{\bullet 1\bullet})$

$$= \text{Var}(\bar{e}_{\bullet 1k}) + \text{Var}(\bar{e}_{\bullet 1\bullet}) - 2\,\text{Cov}(\bar{e}_{\bullet 1k}, \bar{e}_{\bullet 1\bullet})$$

$$= \frac{\sigma_e^2}{3} + \frac{\sigma_e^2}{12} - 2\,\text{Cov}\left(\bar{e}_{\bullet 1k}, \frac{1}{4}(\bar{e}_{\bullet 11} + \bar{e}_{\bullet 12} + \bar{e}_{\bullet 13} + \bar{e}_{\bullet 14})\right)$$

$$= \frac{\sigma_e^2}{3} + \frac{\sigma_e^2}{12} - 2\,\frac{1}{4}\,\text{Var}(\bar{e}_{\bullet 1k}) = \frac{\sigma_e^2}{3} + \frac{\sigma_e^2}{12} - \frac{2\sigma_e^2}{12}$$

$$= \frac{\sigma_e^2}{4}$$

Therefore, the EMS for block×fertilizer is

$$\frac{3}{(4-1)(4-1)} \sum_{k=1}^{4} E\left(\sum_{j=1}^{4} (\bar{e}_{\cdot jk} - \bar{e}_{\cdot j \cdot} - \bar{e}_{\cdot \cdot k} + \bar{e}_{\cdot \cdot \cdot})^2 \right)$$

$$= \frac{1}{3} \sum_{k=1}^{4} (4-1)\, \sigma_e^2/4 = \frac{1}{3}\, 4(4-1)\sigma_e^2/4 = \sigma_e^2 .$$

The EMS for block×genotype×fertilizer can also be shown to be σ_e^2. That is why we combine both of these sums of squares into one sum of squares called "error."

3. This is a split-plot experiment. Cake recipe (CR) is the whole-plot treatment factor and frosting recipe (FR) is the split-plot treatment factor.

We need one random effect for each cake/baker (whole-plot xu) and one random effect for each cake half (split-plot xu). The measurement process adds additional complications and suggests a random effect for each judge.

a)

$$X = \left[\underset{2\times1}{\underset{\sim}{1}} \otimes \underset{4\times4}{I} \otimes \underset{2\times1}{\underset{\sim}{1}} \right] \qquad \underset{\sim}{\beta} = \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{21} \\ \mu_{22} \end{bmatrix}$$

$$Z = [Z_b, Z_h, Z_t] \qquad \underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{b} \\ \underset{\sim}{h} \\ \underset{\sim}{t} \end{bmatrix} \qquad \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$Z_b = \left[\underset{4\times4}{I} \otimes \underset{4\times1}{\underset{\sim}{1}} \right]$$

$$Z_h = \left[\underset{8\times8}{I} \otimes \underset{2\times1}{\underset{\sim}{1}} \right] \qquad \underset{\sim}{h} = \begin{bmatrix} h_{11} \\ h_{12} \\ h_{21} \\ h_{22} \\ h_{31} \\ h_{32} \\ h_{41} \\ h_{42} \end{bmatrix} \qquad \underset{\sim}{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \end{bmatrix}$$

$$Z_t = \underset{2\times2}{I} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) $\begin{bmatrix} \underset{\sim}{u} \\ \underset{\sim}{e} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{b} \\ \underset{\sim}{h} \\ \underset{\sim}{t} \\ \underset{\sim}{e} \end{bmatrix} \sim N\left(\begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \\ \underset{\sim}{0} \\ \underset{\sim}{0} \end{bmatrix}, \begin{bmatrix} \sigma_b^2 I & 0 & 0 & 0 \\ 0 & \sigma_h^2 I & 0 & 0 \\ 0 & 0 & \sigma_t^2 I & 0 \\ 0 & 0 & 0 & \sigma_e^2 I \end{bmatrix} \right).$

4 a) $y_{41} - y_{42} = (\mu + u_4 + e_{41}) - (\mu + u_4 + e_{42})$

$$= e_{41} - e_{42} \sim N(0, 2\sigma_e^2)$$

Thus, $E\left(\frac{1}{2}(y_{41} - y_{42})^2\right) = \frac{1}{2} E(y_{41} - y_{42})^2$

$$= \frac{1}{2} \text{VAR}(y_{41} - y_{42})$$

$$= \frac{1}{2} 2\sigma_e^2 = \sigma_e^2$$

So that $\boxed{\frac{1}{2}(y_{41} - y_{42})^2}$ is an unbiased estimator of σ_e^2.

4 b) $Y_{11}, Y_{21}, Y_{31} \overset{iid}{\sim} N(\mu, \sigma_u^2 + \sigma_e^2)$

Thus, $\dfrac{\sum_{i=1}^{3} (Y_{i1} - \overline{Y}_{\cdot 1})^2}{3-1}$ is an unbiased

estimator of $\sigma_u^2 + \sigma_e^2$, where $\overline{Y}_{\cdot 1} = \frac{1}{3} \sum_{i=1}^{3} Y_{i1}$.

It follows that

$$\frac{\sum_{i=1}^{3} (Y_{i1} - \overline{Y}_{\cdot 1})^2}{2} - \frac{1}{2}(Y_{41} - Y_{42})^2$$ is an unbiased

estimator of σ_u^2.

Many other answers are possible.

5. This is a split-plot experiment. The whole-plot portion of the experiment has a randomized complete block design. Fields are blocks. The four combinations of plant type and fence are the whole-plot treatments. These four treatments are randomly assigned to squares within each field, so the squares of land within each field are the whole-plot experimental units. Each square is split into two rectangles, and the levels of the chemical factor (yes an no) are randomly assigned to rectangles within each square. Thus, rectangles are the split-plot experimental units.

This experiment has the same basic structure as our classic split-plot experiment. The main difference is that the four whole-plot treatments with three degrees of freedom can be partitioned into three one-degree-of-freedom pieces corresponding to plant type, fence, and plant type-by-fence interaction.

The ANOVA table we discussed in class for the classic split-plot experiment still applies.

Source	DF
Block	$b-1$
WP Trt	$w-1$
Block x WP Trt	$(b-1)(w-1)$
SP Trt	$s-1$
WP Trt x SP Trt	$(w-1)(s-1)$
Error	$(b-1)(s-1)+(b-1)(w-1)(s-1)$
C. Total	$bws-1$

This line is Block x SP Trt + Block x WP Trt x SP Trt as discussed in class.

In our special case, this table becomes

Source	DF
field	7
trt	3
field x trt	21
chem	1
trt x chem	3
error	28
c.total	63

The trt line can be partitioned as follows:

plant type	1
fence	1
plant type x fence	1

Likewise, field x trt can be partitioned as

field x plant type	7
field x fence	7
field x plant type x fence	7

As shown in the table of Expected Mean Squares, the model we assume for the data implies that all these three Mean Squares have Expected Mean Square (EMS) equal to $2\sigma_s^2 + \sigma_e^2$. Thus, there is no reason to seperate the field x trt sum of squares into 3 pieces. If we want to estimate $2\sigma_s^2 + \sigma_e^2$, we should use

$$MS_{field x trt} = \frac{SS_{field x planttype} + SS_{field x fence} + SS_{field x planttype x fence}}{df_{field x planttype} + df_{field x fence} + df_{field x planttype x fence}}$$

$$= \frac{SS_{field x trt}}{df_{field x trt}} = \frac{7*16.0 + 7*16.5 + 7*15.5}{7+7+7} = 16$$

Similarly, the last two lines of the ANOVA table on page 14 of these solutions can be partitioned as

$$\boxed{\text{trt} \times \text{chem} \quad 3} \left\{ \begin{array}{ll} \text{plant type} \times \text{chem} & 1 \\ \text{fence} \times \text{chem} & 1 \\ \text{plant type} \times \text{fence} \times \text{chem} & 1 \end{array} \right.$$

$$\boxed{\text{error} \quad 28} \left\{ \begin{array}{ll} \text{field} \times \text{chem} & 7 \\ \boxed{\text{field} \times \text{trt} \times \text{chem} \quad 21} & \left\{ \begin{array}{ll} \text{field} \times \text{pt} \times \text{chem} & 7 \\ \text{field} \times \text{fence} \times \text{chem} & 7 \\ \text{field} \times \text{pt} \times \text{fence} \times \text{chem} & 7 \end{array} \right. \end{array} \right.$$

According to our model, every one of the terms that make up the error has EMS equal to σ_e^2.

Let A = field x chem, B = field x pt x chem, C = field x fence x chem, and D = field x pt x fence x chem.

To estimate σ_e^2 we should use

$$\frac{df_A MS_A + df_B MS_B + df_C MS_C + df_D MS_D}{df_A + df_B + df_C + df_D} = \frac{SS_A + SS_B + SS_C + SS_D}{df_A + df_B + df_C + df_D}.$$

This is just the usual MS_{error} for the ANOVA table on slide 14. In this case, the calculation needed is

$$MS_{error} = \frac{7(2.3) + 7(1.7) + 7(2.1) + 7(1.9)}{28}$$

$$= 2.0.$$

a) $\bar{Y}_{\cdot 1\cdot\cdot} - \bar{Y}_{\cdot 2\cdot\cdot} = \dfrac{18+16+15+14}{4} - \dfrac{12+15+8+7}{4}$

$= 15.75 - 10.5 = 5.25$

b) $Var(\bar{Y}_{\cdot 1\cdot\cdot} - \bar{Y}_{\cdot 2\cdot\cdot}) = Var(\bar{S}_{\cdot 1\cdot} - \bar{S}_{\cdot 2\cdot} + \bar{e}_{\cdot 1\cdot\cdot} - \bar{e}_{\cdot 2\cdot\cdot})$

$= \dfrac{2\sigma_s^2}{16} + \dfrac{2\sigma_e^2}{32}$

$= \dfrac{2\sigma_s^2 + \sigma_e^2}{16}$

$SE(\bar{Y}_{\cdot 1\cdot\cdot} - \bar{Y}_{\cdot 2\cdot\cdot}) = \sqrt{\dfrac{MS_{field \times trt}}{16}} = 1$

c) $\bar{y}_{\cdot 11 \cdot} - \bar{y}_{\cdot 21 \cdot} = \frac{18+16}{2} - \frac{12+15}{2} = 17 - 13.5 = 3.5$

d) $Var(\bar{y}_{\cdot 11 \cdot} - \bar{y}_{\cdot 21 \cdot}) = Var(\bar{s}_{\cdot 1 \cdot} - \bar{s}_{\cdot 2 \cdot} + \bar{e}_{\cdot 11 \cdot} - \bar{e}_{\cdot 21 \cdot})$

$$= \frac{2\sigma_s^2}{16} + \frac{2\sigma_e^2}{16}$$

$$= \frac{\sigma_s^2 + \sigma_e^2}{8} = \frac{1}{8}\left\{\tfrac{1}{2}(2\sigma_s^2 + \sigma_e^2) + \tfrac{1}{2}\sigma_e^2\right\}$$

$$SE(\bar{y}_{\cdot 11 \cdot} - \bar{y}_{\cdot 21 \cdot}) = \sqrt{\frac{1}{8}\left(\frac{1}{2} MS_{field \times trt} + \frac{1}{2} MS_{error}\right)}$$

$$= \sqrt{\frac{1}{8}\left(\frac{1}{2} 16 + \frac{1}{2} 2\right)} = \sqrt{9/8}$$

e) $H_0: \bar{\mu}_{\cdot 11} - \bar{\mu}_{\cdot 21} - \bar{\mu}_{\cdot 12} + \bar{\mu}_{\cdot 22} = 0$

The BLUE of the linear combination above is

$$\bar{Y}_{\cdot 11\cdot} - \bar{Y}_{\cdot 21\cdot} - \bar{Y}_{\cdot 12\cdot} + \bar{Y}_{\cdot 22\cdot} = (17-13.5) - (14.5-7.5)$$

$$= -3.5$$

$$Var(\bar{Y}_{\cdot 11\cdot} - \bar{Y}_{\cdot 21\cdot} - \bar{Y}_{\cdot 12\cdot} + \bar{Y}_{\cdot 22\cdot})$$

$$= Var(\bar{e}_{\cdot 11\cdot} - \bar{e}_{\cdot 21\cdot} - \bar{e}_{\cdot 12\cdot} + \bar{e}_{\cdot 22\cdot})$$

$$= 4 \frac{\sigma_e^2}{16} = \sigma_e^2/4$$

$$t = \frac{-3.5}{\sqrt{MSerror/4}} \Rightarrow F = \frac{(3.5)^2}{MSerror/4} = 2(3.5)^2.$$