510 EXAM 2 SOLUTIONS

SPRING 2015

1a) 4
1b) 8
1c) 7
1d) 8

2a) 6
2b) 7

3a) 6
3b) 6
3c) 8
3d) 8
3e) 6

4a) 9
4b) 4
4c) 9
4d) 4

1a) $\hat{\sigma}_1^2 = 2.0^2 = 4$

$\hat{\sigma}_2^2 = 3.0^2 = 9$

$\hat{\sigma}_3^2 = 1.0^2 = 1$

b) $$\frac{(1/4)(5.5)+(1/9)(6.7)+(1/1)(1.4)}{1/4 + 1/9 + 1/1}$$ ← THIS ANSWER GETS FULL CREDIT

$$= \frac{9(5.5)+4(6.7)+36(1.4)}{49}$$

c) $$\sqrt{\left(\frac{9}{49}\right)^2 4 + \left(\frac{4}{49}\right)^2 9 + \left(\frac{36}{49}\right)^2 1}$$

1d) $Y_1 - Y_2 = u_1 - u_2 + \varepsilon_1 - \varepsilon_2$

$$\Rightarrow \quad Y_1 - Y_2 \sim N(0, 2\sigma_u^2 + 4 + 9)$$

$$\Rightarrow \quad E(Y_1 - Y_2)^2 = 2\sigma_u^2 + 13$$

$$\Rightarrow E\left[\frac{(Y_1 - Y_2)^2 - 13}{2}\right] = \sigma_u^2$$

$\therefore \quad \dfrac{(Y_1 - Y_2)^2 - 13}{2}$ IS AN UNBIASED ESTIMATOR OF σ_u^2.

2a) THIS IS A SPLIT-PLOT EXPERIMENT. CUSTOMERS ARE THE WHOLE-PLOT EXPERIMENTAL UNITS. CUSTOMER VISITS TO THE CHECKOUT PAGE ARE THE SPLIT-PLOT EXPERIMENTAL UNITS.

2b)

SOURCE	DEGREES OF FREEDOM
SETUP	3
CUSTOMER(SETUP)	24,999 x 4 = 99,996
BACKGROUND IMAGE	2
SETUP x BACKGROUND IMAGE	6
ERROR	199,992
C. TOTAL	299,999

3a) $\dfrac{7.4-0.9}{1.6}$

3b) $\bar{Y}_1 - \bar{Y}_2 = \dfrac{Y_{11}+Y_{12}+Y_{13}}{3} - \dfrac{Y_{21}+Y_{24}+Y_{25}}{3}$

$$= \mu_1 + \frac{u_1+u_2+u_3}{3} + \frac{e_{11}+e_{12}+e_{13}}{3}$$

$$-\left(\mu_2 + \frac{u_1+u_4+u_5}{3} + \frac{e_{21}+e_{24}+e_{25}}{3}\right)$$

$$= \mu_1 - \mu_2 + \frac{u_2+u_3-u_4-u_5}{3} + \frac{e_{11}+e_{12}+e_{13}-e_{21}-e_{24}-e_{25}}{3}$$

$$E(\bar{Y}_1 - \bar{Y}_2) = \mu_1 - \mu_2$$

3c) $VAR(\bar{Y}_1 - \bar{Y}_2) = \dfrac{4\sigma_u^2}{9} + \dfrac{6\sigma_e^2}{9} = \dfrac{4}{9}\sigma_u^2 + \dfrac{2}{3}\sigma_e^2$

$$3d)\ \frac{4}{9}\left(\frac{\sigma_e^2 + 1.6\sigma_u^2}{1.6}\right) + \left[\frac{2}{3} - \frac{4}{(9)(1.6)}\right]\sigma_e^2 = \frac{4}{9}\sigma_u^2 + \frac{2}{3}\sigma_e^2$$

$$\frac{4}{9}\,\frac{1}{1.6}\,(7.4) + \left[\frac{2}{3} - \frac{4}{(9)(1.6)}\right](0.9) = \widehat{VAR}(\bar{Y}_1 - \bar{Y}_2)$$

$$3e)\ \text{Let } a_1 = \frac{4}{9}\,\frac{1}{1.6} = \frac{4}{14.4} = \frac{2}{7.2}$$

$$\text{Let } a_2 = \left[\frac{2}{3} - \frac{4}{14.4}\right] = \frac{28.8 - 12}{43.2} = \frac{16.8}{43.2}$$

USING SATTERTHWAITE'S APPROXIMATION,

$$df \cong \frac{(a_1 7.4 + a_2 0.9)^2}{a_1^2\, 7.4^2/5 + a_2^2\, 0.9^2/3}$$

4 a) THIS IS A BALANCED DESIGN. ROWS ARE THE EXPERIMENTAL UNITS FOR THE FERTILIZER FACTOR. COLUMNS ARE THE EXPERIMENTAL UNITS FOR THE PESTICIDE FACTOR. EACH OF THESE EXPERIMENTAL UNITS PROVIDES MORE THAN ONE OBSERVATION. THEREFORE, WE NEED A MODEL THAT INCLUDES A RANDOM EFFECT FOR EACH EXPERIMENTAL UNIT. LET y_{ijkl} BE THE YIELD FOR THE kTH ROW TREATED WITH THE iTH FERTILIZER IN THE lTH COLUMN TREATED WITH THE jTH PESTICIDE. CONSIDER THE MODEL

$$y_{ijkl} = \mu_{ij} + r_{ik} + c_{jl} + e_{ijkl},$$

WHERE $\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}$ ARE UNKNOWN CONSTANTS; $r_{11}, r_{12}, r_{21}, r_{22} \overset{iid}{\sim} N(0, \sigma_r^2)$; $c_{11}, c_{12}, c_{21}, c_{22} \overset{iid}{\sim} N(0, \sigma_c^2)$; AND $e_{ijkl} \overset{ind}{\sim} N(0, \sigma_e^2)$; AND ALL RANDOM TERMS ARE INDEPENDENT.

FOR THIS MODEL, THE FIRST THREE LINES OF THE ANOVA TABLE PROVIDED BY R GIVE THE CORRECT DF, SS, AND MS. THE RESIDUAL LINE NEEDS TO BE FURTHER PARTITIONED.

SOURCE	DF	SS
ROW (FERT)	$(2-1)(2)=2$	$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}(\bar{Y}_{i\cdot k\cdot}-\bar{Y}_{i\cdot\cdot\cdot})^2 = 40$
COL (PEST)	$(2-1)(2)=2$	$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}(\bar{Y}_{\cdot j\cdot l}-\bar{Y}_{\cdot j\cdot\cdot})^2 = 20$
ERROR	$12-2-2=8$	$70-40-20=10$

THE F TEST STATISTIC FOR FERTILIZER MAIN EFFECTS IS

$$\frac{MS_{FERT}}{MS_{ROW(FERT)}} = \frac{36}{40/2} = 1.8.$$

THE DENOMINATOR MS IS ROW(FERT) BECAUSE ROWS NESTED IN FERTILIZER ARE THE EXPERIMENTAL UNITS FOR THE FERTILIZER TREATMENT.

4b) DF ARE 1 FOR NUMERATOR AND 2 FOR DENOMINATOR

4c) THE SS FOR INTERACTION BETWEEN FERT. AND PEST. IS

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}\left(\bar{Y}_{ij..} - \bar{Y}_{i...} - \bar{Y}_{.j..} + \bar{Y}_{....}\right)^2$$

$$= 4\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..} + \bar{e}_{ij..} - \bar{e}_{i...} - \bar{e}_{.j..} + \bar{e}_{....}\right)^2$$

$$\therefore\ E\left(MS_{F:P}\right) = 4\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}\right)^2 + 4\sum_{j=1}^{2} E\left(\sum_{i=1}^{2}\left(\bar{e}_{ij..} - \bar{e}_{i...} - \bar{e}_{.j..} + \bar{e}_{....}\right)^2\right)$$

$$= 4\sum\sum\left(\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}\right)^2 + (4)(2)(2-1)\,VAR\left(\bar{e}_{11..} - \bar{e}_{1...}\right)$$

NOW $VAR\left(\bar{e}_{11..} - \bar{e}_{1...}\right) = \dfrac{\sigma_e^2}{4} + \dfrac{\sigma_e^2}{8} - 2\dfrac{\sigma_e^2}{8} = \dfrac{\sigma_e^2}{8}$

THUS, $E\left(MS_{F:P}\right) = 4\sum\sum\left(\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}\right)^2 + \sigma_e^2$, $E(MS_{ERROR})$

AND WE SEE THAT $MS_{ERROR} = 10/8$ IS THE APPROPRIATE DENOMINATOR FOR THE F-TEST FOR INTERACTION.

$$F = \frac{4}{10/8} = \frac{32}{10} = 3.2$$

4d) DF = 1 AND 8.