STAT 510

EXAM 2 SOLUTIONS

SPRING 2018

POINTS POSSIBLE: 100

1a) 8	2a) 10	3a) 10	4a) 10
1b) 10	2b) 8	3b) 8	4b) 10
1c) 8	2c) 8		4c) 10

1a) $$\mathrm{Cov}(y_{1111}, y_{1221}) = \mathrm{cov}(m_{11} + e_{1111}, m_{11} + e_{1221})$$
$$= \mathrm{Cov}(m_{11}, m_{11})$$
$$= \mathrm{Var}(m_{11})$$
$$= \sigma_m^2$$

$$\mathrm{Var}(y_{ijkl}) = \mathrm{Var}(m_{il} + e_{ijkl}) = \sigma_m^2 + \sigma_e^2$$

THUS, $$\mathrm{Corr}(y_{1111}, y_{1221}) = \frac{\sigma_m^2}{\sigma_m^2 + \sigma_e^2}$$

1b) LET M = MATTRESS TYPE
L = LIGHT
S = SOUND

SOURCE	DF	
M	2	
man(M)	15	← WHOLE-PLOT ERROR
L	1	
S	1	
L×S	1	
M×L	2	
M×S	2	
M×L×S	2	
SPLIT-PLOT ERROR	45	← L×man(M) + S×man(M) + L×S×man(M)
C. TOTAL	71	

1c) $X = \underset{3\times 3}{I} \otimes \underset{6\times 1}{\underset{\sim}{1}} \otimes \underset{4\times 4}{I}$

$Z = \underset{18\times 18}{I} \otimes \underset{4\times 1}{\underset{\sim}{1}}$

2 a) $\underset{\sim}{y}'(P_{[\underset{\sim}{1},A]} - P_{\underset{\sim}{1}})\underset{\sim}{y} = \| \hat{\underset{\sim}{y}}_{[\underset{\sim}{1},A]} - \hat{\underset{\sim}{y}}_{\underset{\sim}{1}} \|^2$

AVERAGE OF y VALUES FOR $A = 1$ IS $\dfrac{2\times 3.0 + 8\times 5.0}{10} = 4.6$

AVERAGE OF y VALUES FOR $A = 2$ IS $\dfrac{6\times 7.0 + 4\times 3.0}{10} = 5.4$

AVERAGE OF ALL y VALUES IS $\dfrac{4.6 + 5.4}{2} = 5.0$

THUS, $\| \hat{\underset{\sim}{y}}_{[\underset{\sim}{1},A]} - \hat{\underset{\sim}{y}}_{\underset{\sim}{1}} \|^2 = 10(4.6-5.0)^2 + 10(5.4-5.0)^2$

$= 10(.4)^2 + 10(.4)^2$

$= 20(.4)^2 = 20(.16)$

$= 3.2$

2 b)

$$
\begin{array}{cc|c|c|c}
 & & \multicolumn{2}{c|}{B} & \\
 & & 1 & 2 & \\
\hline
\multirow{2}{*}{A} & 1 & \mu_{11} & \mu_{12} & \bar{\mu}_{1.} \\
\cline{2-5}
 & 2 & \mu_{21} & \mu_{22} & \bar{\mu}_{2.} \\
\end{array}
$$

LSMEAN FOR A2 IS BLUE OF

$$\bar{\mu}_{2.} = \frac{\mu_{21} + \mu_{22}}{2}$$

$$\hat{\bar{\mu}}_{2.} = \frac{\hat{\mu}_{21} + \hat{\mu}_{22}}{2} = \frac{\bar{Y}_{21.} + \bar{Y}_{22.}}{2}$$

$$= \frac{7.0 + 3.0}{2} = 5.0$$

2c) LET $X = \begin{bmatrix} \underset{2\times1}{\underset{\sim}{1}} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{8\times1}{\underset{\sim}{1}} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{6\times1}{\underset{\sim}{1}} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{4\times1}{\underset{\sim}{1}} \end{bmatrix}$ $\quad \underset{\sim}{\beta} = \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{21} \\ \mu_{22} \end{bmatrix}$

$$(X'X)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/8 & 0 & 0 \\ 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix} \qquad \begin{aligned} \underset{\sim}{c}_1' &= [1 \quad 1 \quad -1 \quad -1] \\ \underset{\sim}{c}_2' &= [1 \quad -1 \quad 1 \quad -1] \end{aligned}$$

$$\underset{\sim}{c}_1'(X'X)^{-1}\underset{\sim}{c}_2 = [1/2, 1/8, -1/6, -1/4] \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= 1/2 - 1/8 - 1/6 + 1/4$$

$$= \frac{12 - 3 - 4 + 6}{24} = \frac{11}{24} \neq 0$$

NOT ORTHOGONAL CONTRASTS

3a) $(\underset{\sim}{I} + a\underset{\sim}{1}\underset{\sim}{1}')' = \underset{\sim}{I}' + a(\underset{\sim}{1}\underset{\sim}{1}')' = \underset{\sim}{I} + a\underset{\sim}{1}\underset{\sim}{1}'$;

SO $\underset{\sim}{I} + a\underset{\sim}{1}\underset{\sim}{1}'$ IS SYMMETRIC. NOW SUPPOSE $x \in \mathbb{R}^n \setminus \{0\}$.

THEN
$$\underset{\sim}{x}' V \underset{\sim}{x} = \underset{\sim}{x}'(\underset{\sim}{I} + a\underset{\sim}{1}\underset{\sim}{1}')\underset{\sim}{x}$$
$$= \underset{\sim}{x}'\underset{\sim}{x} + a\underset{\sim}{x}'\underset{\sim}{1}\underset{\sim}{1}'\underset{\sim}{x}$$
$$= \sum_{i=1}^{n} x_i^2 + a\left(\sum_{i=1}^{n} x_i\right)^2 \leftarrow \left(\begin{array}{l}\geq 0 \text{ BECAUSE} \\ a > 0\end{array}\right)$$
$$\geq \sum_{i=1}^{n} x_i^2 > 0$$

(BECAUSE $\underset{\sim}{x} \neq \underset{\sim}{0} \Rightarrow x_i^2 > 0$ FOR AT LEAST ONE $i \in \{1, \dots, n\}$.)

3b)
$$V X = \left[\underset{n\times n}{I} + a\, \underset{\sim}{1}_n \underset{\sim}{1}_n'\right] X$$
$$= X + a\, \underset{\sim}{1}_n \underset{\sim}{1}_n' X$$
$$= X + a\, X \underset{\sim}{1}_p \underset{\sim}{1}_n' X \quad (\underset{\sim}{1}_n = X \underset{\sim}{1}_p)$$
$$= X\left[\underset{p\times p}{I} + a\, \underset{\sim}{1}_p \underset{\sim}{1}_n' X\right]$$
$$= X\,Q, \text{ WHERE } Q = \left[\underset{p\times p}{I} + a\, \underset{\sim}{1}_p \underset{\sim}{1}_n' X\right]$$

$\therefore$ OLS OF ESTIMABLE $C\underset{\sim}{\beta}$ IS BLUE OF $C\underset{\sim}{\beta}$.

4 a)

THESE DIFFERENT FROM THESE

$$\frac{2S_1 + 2S_2 + S_3 + 2S_4 + S_5 + 8\,e's \;-\; 2S_1 - 2S_2 - S_3 - S_4 - 2S_5 - 8\,e's}{8}$$

$$= \frac{S_4 - S_5 + 8\,e's - 8\,e's}{8}$$

VARIANCE OF ABOVE IS

$$\frac{1}{8^2}\left[2\sigma_s^2 + 16\sigma_e^2\right] = \frac{\sigma_s^2}{32} + \frac{\sigma_e^2}{4}$$

4 b)

SUBJECT	VARIANCE OF DIFF. OF TRT. AVERAGES
1	$Var(\bar{e}_{11\cdot} - \bar{e}_{12\cdot}) = \sigma_e^2\left[\frac{1}{2} + \frac{1}{2}\right] = \sigma_e^2$
2	$Var(\bar{e}_{21\cdot} - \bar{e}_{22\cdot}) = \sigma_e^2\left[\frac{1}{2} + \frac{1}{2}\right] = \sigma_e^2$
3	$Var(e_{311} - e_{321}) = 2\sigma_e^2$
4	$Var(\bar{e}_{41\cdot} - e_{421}) = \sigma_e^2\left[\frac{1}{2} + 1\right]$
5	$Var(e_{511} - \bar{e}_{52\cdot}) = \sigma_e^2\left[1 + \frac{1}{2}\right]$

THUS, VARIANCE OF AVERAGE OF DIFFERENCES IS

$$\left(\frac{1}{5}\right)^2\left[\sigma_e^2 + \sigma_e^2 + 2\sigma_e^2 + 1.5\sigma_e^2 + 1.5\sigma_e^2\right] = \frac{7\sigma_e^2}{25}$$

4 c) WE KNOW THAT INVERSE VARIANCE WEIGHTING IS THE BEST WAY TO COMBINE MULTIPCE INDEPENDENT UNBIASED ESTIMATORS. SO, WE SHOULD WEIGHT THE DIFFENCES OF TREATMENT AVERAGES AS FOLLOWS

$$\frac{1 \times 8 + 1 \times 15 + \frac{1}{2} \times 7 + \frac{1}{1.5} \times 1 + \frac{1}{1.5} \times (-2)}{1 + 1 + \frac{1}{2} + \frac{1}{1.5} + \frac{1}{1.5}}$$

$$= \frac{48 + 90 + 21 + 4 - 8}{6 + 6 + 3 + 4 + 4}$$

$$= \frac{155}{23}$$

THE KEY HERE IS THAT THE SUBJECT-SPECIFIC DIFFERENCES ARE INDEPENDENT OF EACH OTHER. MANY STUDENTS TRIED TO APPLY INVERSE VARIANCE WEIGHTING TO THE ANSWERS FROM (a) AND (b), BUT THESE ESTIMATORS ARE NOT INDEPENDENT. FURTHERMORE, THE BEST WAY TO COMBINE THEM WOULD DEPEND ON THE VALUES OF UNKNOWN VARIANCE COMPONENTS. THE ANSWER ABOVE TO (c) IMPROVES ON THE ANSWER IN (b) NO MATTER WHAT THE VALUES OF σ_s^2 AND σ_e^2 ARE. SEE THE NEXT TWO PAGES FOR SOME MORE COMMENTS ON THIS PROBLEM.

IF $\hat{\Theta}_1, \ldots, \hat{\Theta}_n$ ARE INDEPENDENT UNBIASED ESTIMATORS OF A PARAMETER Θ WITH VARIANCES $V_1, \ldots, V_n$, RESPECTIVELY, THEN

$$\mathrm{VAR}\left(\sum_{i=1}^{n} \frac{1}{V_i} \hat{\Theta}_i \Big/ \sum_{i=1}^{n} 1/V_i \right)$$

$$= \left(\frac{1}{\sum_{i=1}^{n} 1/V_i} \right)^2 \sum_{i=1}^{n} \left(\frac{1}{V_i} \right)^2 V_i$$

$$= \frac{1}{\left(\sum_{i=1}^{n} 1/V_i \right)^2} \sum_{i=1}^{n} 1/V_i = \frac{1}{\sum_{i=1}^{n} 1/V_i}$$

THUS, VARIANCE OF ESTIMATOR IN 4(c) IS

$$\sigma_e^2 \frac{1}{1 + 1 + 1/2 + 1/1.5 + 1/1.5} = \frac{6\sigma_e^2}{6+6+3+4+4}$$

$$= \frac{6\sigma_e^2}{23}.$$

THIS IS INDEED LESS THAN VARIANCE OF $\frac{7\sigma_e^2}{25}$ IN PART (b).

HOWEVER, THE ESTIMATOR DEFINED IN PART (c) IS NOT THE BLUE. THIS CAN BE SEEN BY NOTING THAT THE VARIANCE OF THE ESTIMATOR IN PART (a)

$$\frac{\sigma_s^2}{32} + \frac{\sigma_e^2}{4}$$ COULD BE LESS THAN

$$\frac{6\sigma_e^2}{23}$$ IF σ_s^2 IS SUFFICIENTLY SMALL. AS σ_s^2 APPROACHES 0, THE ESTIMATOR IN PART (a) BECOMES MORE SIMILAR TO THE BLUE. AT $\sigma_s^2 = 0$, THE PART (a) ESTIMATOR IS THE BLUE.