EXAM 2 SOLUTIONS

SPRING 2019

PROBLEM	POINTS POSSIBLE
1.	10
2.	12
3. a)	20
3. b)	6
4.	18
5. a)	8
5. b)	10
5. c)	16

1. Here Are Two Proofs, Starting With The Simplest.

- H Orthogonal $\Rightarrow H'H = I$.

$$\underset{\sim}{x} \neq \underset{\sim}{0} \Rightarrow H'H\underset{\sim}{x} \neq \underset{\sim}{0} \Rightarrow H\underset{\sim}{x} \neq \underset{\sim}{0}. \quad \blacksquare$$

- Let $\underset{\sim}{z} = H\underset{\sim}{x}$. Then

$$\underset{\sim}{z}'\underset{\sim}{z} = (H\underset{\sim}{x})'H\underset{\sim}{x} = \underset{\sim}{x}'H'H\underset{\sim}{x}$$

$$= \underset{\sim}{x}' I \underset{\sim}{x} \quad \text{(By Orthogonality of } H\text{)}$$

$$= \underset{\sim}{x}'\underset{\sim}{x}$$

$$= \sum_{i=1}^{n} x_i^2 > 0$$

(Because $\underset{\sim}{x} \neq \underset{\sim}{0}$ Implies $x_i \neq 0$ For At Least One $i \in \{1, \ldots, n\}$.)

$$\underset{\sim}{z}'\underset{\sim}{z} > 0 \Rightarrow \underset{\sim}{z} \neq \underset{\sim}{0}$$

$$\Rightarrow H\underset{\sim}{x} \neq \underset{\sim}{0} \quad \blacksquare$$

2. $\begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \end{bmatrix} = \underset{\sim}{1}\mu + \underset{\sim}{\varepsilon}$, WHERE

$$\underset{\sim}{\varepsilon} \sim N\left(\underset{\sim}{0}, \sigma^2 \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}\right).$$

THIS IS A SPECIAL CASE OF THE AITKEN MODEL WITH $\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \end{bmatrix}$,

$X = \underset{\sim}{1}_{(n_1+n_2)\times 1}$, AND

$V = \begin{bmatrix} V_1 & O_{n_1 \times n_2} \\ O_{n_2 \times n_1} & V_2 \end{bmatrix}$. THUS, THE BLUE OF μ IS THE GLS ESTIMATOR $(X'V^{-1}X)^{-1}X'V^{-1}\underset{\sim}{y}$.

NOTE THAT $V^{-1} = \begin{bmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{bmatrix}$.

ALSO,

$$X'V^{-1}X = \left[\underset{\sim}{1}'_{n_1 \times 1}, \underset{\sim}{1}'_{n_2 \times 1} \right] \begin{bmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{1}_{n_1 \times 1} \\ \underset{\sim}{1}_{n_2 \times 1} \end{bmatrix}$$

$$= \underset{\sim}{1}'V_1^{-1}\underset{\sim}{1} + \underset{\sim}{1}'V_2^{-1}\underset{\sim}{1}$$

SO THAT $(X'V^{-1}X)^{-1} = \dfrac{1}{\underset{\sim}{1}'V_1^{-1}\underset{\sim}{1} + \underset{\sim}{1}'V_2^{-1}\underset{\sim}{1}}$.

FURTHERMORE,

$$X'V^{-1}y = \underset{\sim}{1}'V_1^{-1}y_1 + \underset{\sim}{1}'V_2^{-1}y_2.$$

THUS, THE BLUE OF μ IS

$$(X'V^{-1}X)^{-1}X'V^{-1}y = \frac{\underset{\sim}{1}'V_1^{-1}y_1 + \underset{\sim}{1}'V_2^{-1}y_2}{\underset{\sim}{1}'V_1^{-1}\underset{\sim}{1} + \underset{\sim}{1}'V_2^{-1}\underset{\sim}{1}}.$$

3. a)

		Dose	
		0	10
Drug	1	6, 2	12, 6
	2	4	16, 10

$$\text{Overall Average} = \frac{6+2+12+6+4+16+10}{7} = 8$$

$$SS_{C.TOTAL} = (6-8)^2 + (2-8)^2 + (12-8)^2 + (6-8)^2 + (4-8)^2 + (16-8)^2 + (10-8)^2$$

$$= 4 + 36 + 16 + 4 + 16 + 64 + 4$$

$$= 144$$

$$\text{Drug 1 Average} = \frac{6+2+12+6}{4} = 6.5$$

$$\text{Drug 2 Average} = \frac{4+16+10}{3} = 10$$

$$SS_{DRUG} = y'(P_{[\underline{1}, DRUG]} - P_{\underline{1}})\, y$$

$$= \| P_{[\underline{1}, DRUG]}\, y - P_{\underline{1}}\, y \|^2$$

$$= 4(6.5-8)^2 + 3(10-8)^2 = 9 + 12 = 21$$

3.a) (CONTINUED)

$$SSE = (6-4)^2 + (2-4)^2 + (4-4)^2 + (12-9)^2 + (6-9)^2 + (16-13)^2 + (10-13)^2$$

$$= 4+4+9+9+0+9+9$$

$$= 44$$

$$SS_{DRUG \times DOSE} = (C\hat{\underline{\beta}})'[C(X'X)^{-1}C']^{-1}C\hat{\underline{\beta}}$$

$$= \frac{(4-9-4+13)^2}{\frac{1}{2}+\frac{1}{2}+\frac{1}{1}+\frac{1}{2}}$$

$$= \frac{2 \times 16}{5} = 6.4$$

$$SS_{DRUG} + SS_{DOSE} + SS_{DRUG:DOSE} + SSE = SS_{C.TOTAL}$$

$$\Longrightarrow SS_{DOSE} = SS_{C.TOTAL} - SS_{DRUG} - SS_{DRUG:DOSE} - SSE$$

$$= 144 - 21 - 6.4 - 44$$

$$= 72.6$$

3 b) THE FOUR TREATMENTS IN THIS EXPERIMENT ARE

0 mg OF DRUG 1

0 mg OF DRUG 2

10 mg OF DRUG 1

10 mg OF DRUG 2

NOTE THAT IF THE DOSE IS 0 mg, THE DRUG (REGARDLESS OF 1 OR 2) IS NOT ADMINISTERED. THUS, THE TREATMENTS 0 mg OF DRUG 1 AND 0 mg OF DRUG 2 ARE IDENTICAL. THUS, WE SHOULD FIT A MODEL WITH JUST THREE TREATMENT GROUPS RATHER THAN FOUR.

TREATMENT	DESCRIPTION
1	NO DRUG
2	10 mg OF DRUG 1
3	10 mg OF DRUG 2

$$y_{ij} = \mu_i + \epsilon_{ij}$$

$$\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$$

$i = 1, 2, 3$ $\quad n_1 = 3$

$\quad n_2 = 2$

$j = 1, \ldots, n_i$ $\quad n_3 = 2$

4.

Source	DF
Brew Time	1
Morning(Brew Time)	12
Additive	1
Brew Time x Additive	1
Error	12
C. Total	27

THE EXACT WORDING IS NOT IMPORTANT. FOR EXAMPLE, MORNING (BREW TIME) COULD ALTERNATIVELY BE POT(BREW TIME). ERROR COULD BE SPLIT-PLOT ERROR OR ADDITIVE x POT (BREW TIME). EITHER WAY, THIS LINE CORRESPONDS TO CUPS OF COFFEE.

NOTE THAT ALTHOUGH THE NUMBERS DIFFER, THIS EXPERIMENT IS LIKE THE DIET DRUG SPLIT-PLOT EXPERIMENT FROM OUR NOTES.

5. a) BASED ON THE PROVIDED EXPECTED MEAN SQUARES, IT IS EASY TO SEE THAT

$$\frac{MS_T - MS_{T \times SM}}{6} = \frac{17.9 - 5.3}{6} = 2.1$$

IS THE VALUE OF AN UNBIASED ESTIMATOR FOR σ_t^2.

ANOTHER OPTION WOULD BE

$$\frac{MS_T - M_{T \times SM} + MS_{T \times G} - MS_{T \times SM \times G}}{6},$$

BUT THIS CLEARLY HAS GREATER VARIANCE THAN $\frac{MS_T - MS_{T \times SM}}{6}$.

5 b) COURSE NOTES ON ANOVA ANALYSIS OF SPLIT-PLOT EXPERIMENTS EXPLAIN THAT DF-WEIGHTED COMBINATION OF MEAN SQUARES WITH THE SAME EXPECTATION RESULTS IN THE LOWEST VARIANCE UNBIASED ESTIMATOR. THUS, σ_e^2 IS BEST ESTIMATED BY

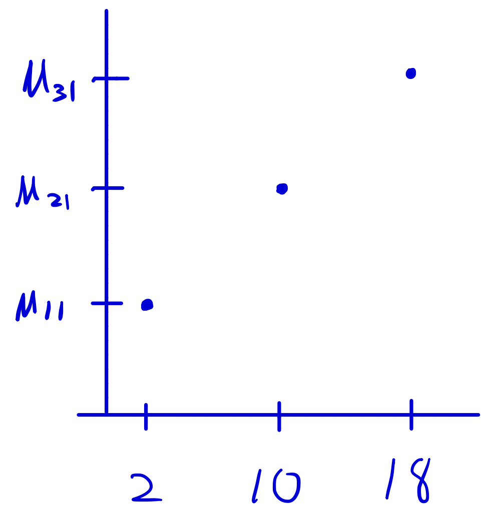
$$\frac{DF_{T\times G}\, MS_{T\times G} + DF_{T\times SM\times G}\, MS_{T\times SM\times G}}{DF_{T\times G} + DF_{T\times SM\times G}}$$

$$= \frac{7\times 3.3 + 14\times 3.9}{21}$$

$$= \frac{1}{3}\,3.3 + \frac{2}{3}\,3.9$$

$$= 3.7$$

5c) AS WE HAVE SEEN IN PREVIOUS EXAMPLES WITH EQUALLY SPACED QUANTITATIVE LEVELS, WE NEED TO TEST



$$H_0: \mu_{21} - \mu_{11} = \mu_{31} - \mu_{21}$$

WHICH IS EQUIVALENT TO

$$H_0: \mu_{11} - 2\mu_{21} + \mu_{31} = 0$$

BECAUSE WE HAVE A BALANCED SPLIT-PLOT EXPERIMENT (LIKE THE FIELD SPLIT-PLOT EXPERIMENT IN OUR COURSE NOTES),

THE BLUE OF μ_{ij} IS $\overline{Y}_{ij\cdot}$ $\forall\, i, j$,

AND IT FOLLOWS THAT

THE BLUE OF $\mu_{11} - 2\mu_{21} + \mu_{31}$ IS

$$\overline{Y}_{11\cdot} - 2\overline{Y}_{21\cdot} + \overline{Y}_{31\cdot} = \mu_{11} - 2\mu_{21} + \mu_{31}$$
$$+ \overline{P}_{1\cdot} - 2\overline{P}_{2\cdot} + \overline{P}_{3\cdot}$$
$$+ \overline{e}_{11\cdot} - 2\overline{e}_{21\cdot} + \overline{e}_{31\cdot}$$

5c) (CONTINUED)

THUS, $VAR(\bar{Y}_{11\cdot} - 2\bar{Y}_{21\cdot} + \bar{Y}_{31\cdot})$

$$= \sigma_p^2/8 + 4\sigma_p^2/8 + \sigma_p^2/8$$
$$+ \sigma_e^2/8 + 4\sigma_e^2/8 + \sigma_e^2/8$$
$$= \frac{3}{4}(\sigma_p^2 + \sigma_e^2)$$

THE VALUE OF AN UNBIASED ESTIMATOR FOR $\sigma_p^2 + \sigma_e^2$ IS

$$\tfrac{1}{2} MS_{Txsm} + \tfrac{1}{2}\, 3.7 = (5.3 + 3.7)/2 = 4.5$$

BECAUSE $E(MS_{Txsm}) = 2\sigma_p^2 + \sigma_e^2$ AND BECAUSE 3.7 IS THE VALUE OF AN UNBIASED ESTIMATOR FOR σ_e^2 BY PART (b).

THUS, $$t = \frac{4.2 - 2 \times 6.3 + 9.4}{\sqrt{(3/4)\, 4.5}} = \frac{2}{\sqrt{13.5}}$$