EXAM 2 SOLUTIONS

SPRING 2020

1. a)	6
b)	8
c)	8
d)	8
2. a)	15
b)	10
c)	10
3.	15
4. a)	10
4 b)	10

POINTS POSSIBLE

1 a) $X = \underset{2\times 2}{I} \otimes \underset{\sim}{1}_{28\times 1}$

$Z = \underset{14\times 14}{I} \otimes \underset{\sim}{1}_{4\times 1}$

b)

SOURCE	DF	
RECIPE	2-1	= 1
BATCH (RECIPE)	(7-1)(2)	= 12
BOWL (BATCH, RECIPE)	(4-1)(7×2)	= 42
C. TOTAL		55

c) $(j-1)(i) = ij - i$

$$\sum_{i=1}^{2}\sum_{j=1}^{7}\sum_{k=1}^{4}\left(\bar{Y}_{ij.} - \bar{Y}_{i..}\right)^2 = 4\sum_{i=1}^{2}\sum_{j=1}^{7}\left(\bar{Y}_{ij.} - \bar{Y}_{i..}\right)^2$$

d) BATCHES ARE THE EXPERIMENTAL UNITS FOR RECIPES. THUS, BATCH (RECIPE) IS THE CORRECT ERROR LINE OF THE ANOVA TABLE. SO DF = 12.

2 a) $Y_{prt} = \mu + b_p + \alpha_r + \beta_t + \gamma_{rt} + w_{pr} + e_{prt}$,

WHERE $\mu, \alpha_r, \beta_t, \gamma_{rt}$ ARE UNKNOWN FIXED PARAMETERS,

$b_p \overset{iid}{\sim} N(0, \sigma_b^2)$ (RANDOM BLOCK EFFECTS FOR PEOPLE)

$w_{rt} \overset{iid}{\sim} N(0, \sigma_w^2)$ (RANDOM WHOLE-PLOT EFFECTS FOR BATCHES)

$e_{prt} \overset{iid}{\sim} N(0, \sigma_e^2)$ (RANDOM ERRORS FOR BOWLS)

ALL RANDOM EFFECTS ABOVE ARE INDEPENDENT.

2b) $\bar{y}_{\cdot r \cdot} = \mu + \bar{b}_{\cdot} + \alpha_r + \bar{\beta}_{\cdot} + \bar{\gamma}_{r\cdot} + \bar{w}_{\cdot r} + \bar{e}_{\cdot r \cdot}$

$$\text{VAR}(\bar{y}_{\cdot r \cdot}) = \text{VAR}(\bar{b}_{\cdot} + \bar{w}_{\cdot r} + \bar{e}_{\cdot r \cdot})$$

$$= \frac{\sigma_b^2}{7} + \frac{\sigma_w^2}{7} + \frac{\sigma_e^2}{28}$$

2c)

Source	DF	
Person	$7-1$	$= 6$
Recipe	$2-1$	$= 1$
Person x Recipe	$(7-1)(2-1)$	$= 6$
Topping	$4-1$	$= 3$
Recipe x Topping	$(2-1)(4-1)$	$= 3$
Error	$(7-1)(4-1)+(7-1)(2-1)(4-1)$	$= 36$
C. Total		55

3. REGARDLESS OF THE ORDER THAT FACTOR A & B ENTER THE MODEL, THE FULL MODEL IS THE CELL MEANS MODEL. WE KNOW SS A*B, SSE, AND SS C.TOTAL ARE EXACTLY THE SAME. THUS, WE KNOW RIGHT AWAY

SOURCE	SUM OF SQUARES
B	
A	
A*B	5.415
ERROR	1.250
C.TOTAL	43.880

NOW SSB IN THIS TABLE IS

$$SS(B|1) = \underset{\sim}{y}'(P_{[\underline{1},B]} - P_{\underline{1}})\underset{\sim}{y}$$

$$= \| P_{[\underline{1},B]}\underset{\sim}{y} - P_{\underline{1}}\underset{\sim}{y} \|^2$$

3. (CONTINUED)

TO GET $P_{[\underset{\sim}{1},B]}\, \underset{\sim}{y}$, NOTE THIS IS JUST THE VECTOR OF FITTED VALUES FOR THE MODEL WITH JUST TWO TREATMENT GROUPS DEFINED BY THE LEVELS OF FACTOR B. THE AVERAGE OF RESPONSE VALUES FOR B=1 IS

$$\frac{4 \times 4.50 + 2 \times 9.00}{6} = 6.0$$

FOR B=2, THE AVERAGE IS

$$\frac{2 \times 6.90 + 4 \times 8.55}{6} = 8.0$$

THE AVERAGE OF ALL RESPONSE VALUES IS

$$\frac{6 \times 6.0 + 6 \times 8.0}{12} = 7.0 \quad \text{SO} \quad P_{\underset{\sim}{1}}\, \underset{\sim}{y} = 7.0\, \underset{\sim}{1}_{12 \times 1}$$

IT FOLLOWS THAT $\| P_{[\underset{\sim}{1},B]}\, \underset{\sim}{y} - P_{\underset{\sim}{1}}\, \underset{\sim}{y} \|^2$

3. (CONTINUED)

IT FOLLOWS THAT

$$\| P_{[\underline{1},B]}\, \underline{y} - P_{\underline{1}}\, \underline{y} \|^2 = 6(6.0-7.0)^2 + 6(8.0-7.0)^2$$

$$= 12.0$$

SO WE HAVE

SOURCE	SUM OF SQUARES
B	12.000
A	
A*B	5.415
ERROR	1.250
C.TOTAL	43.880

THE REMAINING ENTRY MUST BE

$$43.880 - (12.000 + 5.415 + 1.250)$$

$$= 25.215$$

4. FOR LAB i,

$$\widehat{\tau_1 - \tau_2} = \bar{y}_{i1\cdot} - \bar{y}_{i2\cdot} \quad \text{AND}$$

$$\mathrm{VAR}(\bar{y}_{i1\cdot} - \bar{y}_{i2\cdot}) = \frac{\sigma_i^2}{5} + \frac{\sigma_i^2}{5}$$

$$= .4\sigma_i^2$$

THE BLUE OF $\tau_1 - \tau_2$ IS THE GLS ESTIMATOR, WHICH USES INVERSE VARIANCE WEIGHTING:

$$\frac{\sum_{i=1}^{3} \frac{1}{.4\sigma_i^2}(\bar{y}_{i1\cdot} - \bar{y}_{i2\cdot})}{\sum_{i=1}^{3} 1/.4\sigma_i^2}$$

$$= \sum_{i=1}^{3} \frac{1/\sigma_i^2}{(1/\sigma_1^2 + 1/\sigma_2^2 + 1/\sigma_3^2)} (\bar{y}_{i1\cdot} - \bar{y}_{i2\cdot})$$

4. (CONTINUED)

THE VARIANCE OF THIS ESTIMATOR IS

$$\sum_{i=1}^{3}\left(\frac{1/\sigma_i^2}{1/\sigma_1^2+1/\sigma_2^2+1/\sigma_3^2}\right)^2 .4\,\sigma_i^2$$

$$= \frac{.4\sum_{i=1}^{3} \sigma_i^2/\sigma_i^4}{\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}+\frac{1}{\sigma_3^2}\right)^2}$$

$$= \frac{.4\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}+\frac{1}{\sigma_3^2}\right)}{\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}+\frac{1}{\sigma_3^2}\right)^2}$$

$$= \frac{.4}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}+\frac{1}{\sigma_3^2}}$$

$$\equiv VAR$$

4 a) TO APPROXIMATE THE BLUE, WE REPLACE σ_i^2 WITH MSE_i FOR $i=1,2,3$;

$$\frac{\frac{1}{9.1}5.6 + \frac{1}{15.4}(-3.2) + \frac{1}{6.2}1.3}{\frac{1}{9.1} + \frac{1}{15.4} + \frac{1}{6.2}}$$

$$\simeq 1.84$$

4 b) $SE = \sqrt{\hat{VAR}}$

$$= \sqrt{\frac{.4}{\frac{1}{9.1} + \frac{1}{15.4} + \frac{1}{6.2}}}$$

$$\approx \sqrt{1.19}$$

$$\approx 1.09$$