STAT 510 Homework 2

Due Date: 11:00 A.M., Wednesday, January 29

1. The main purpose of this problem is to derive several results about orthogonal-projection matrices stated near the beginning of slide set 2.

   (a) Use the definition of generalized inverse and the result of problem 7 on Homework 1 to prove that
   $$\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{X} = \boldsymbol{X}$$
   for any $(\boldsymbol{X}'\boldsymbol{X})^{-}$ a generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$.

   (b) Prove that if $\boldsymbol{A}$ is any symmetric matrix and $\boldsymbol{G}$ is any generalized inverse of $\boldsymbol{A}$, then it must be true that $\boldsymbol{G}'$ is also a generalized inverse of $\boldsymbol{A}$.

   (c) Use the results of problems (a) and (b) to prove that
   $$\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}' = \boldsymbol{X}'$$
   for any $(\boldsymbol{X}'\boldsymbol{X})^{-}$ a generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$.

   (d) Show that idempotency of $\boldsymbol{P_X}$ (i.e., $\boldsymbol{P_X}\boldsymbol{P_X} = \boldsymbol{P_X}$) follows from the result of part (a) and, alternatively, from the result of part (c).

   (e) Use parts (a) and (c) to prove that
   $$\boldsymbol{X}\boldsymbol{G}_1\boldsymbol{X}' = \boldsymbol{X}\boldsymbol{G}_2\boldsymbol{X}'$$
   for any two generalized inverses of $\boldsymbol{X}'\boldsymbol{X}$ denoted by $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$. (This says that $\boldsymbol{P_X}$ is the same matrix no matter which generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$ is used to compute it.)

   (f) Use (b) and (e) to prove that $\boldsymbol{P_X}$ is symmetric (i.e., $\boldsymbol{P}'_{\boldsymbol{X}} = \boldsymbol{P_X}$).

2. Suppose $\boldsymbol{X}$ is an $n \times p$ matrix and $\boldsymbol{y}$ is an $n \times 1$ vector. Suppose $\boldsymbol{z} \in \mathcal{C}(\boldsymbol{X})$ and $\boldsymbol{z} \neq \boldsymbol{P_X}\boldsymbol{y}$. Prove that $\|\boldsymbol{y}-\boldsymbol{z}\| > \|\boldsymbol{y}-\boldsymbol{P_X}\boldsymbol{y}\|$. Hint: Note that for any vector $\boldsymbol{a}$ and any vector $\boldsymbol{b} \neq \boldsymbol{0}$ such that $\boldsymbol{a}'\boldsymbol{b} = 0$
   $$\begin{aligned}
   \|\boldsymbol{a}+\boldsymbol{b}\|^2 &= (\boldsymbol{a}+\boldsymbol{b})'(\boldsymbol{a}+\boldsymbol{b}) = (\boldsymbol{a}'+\boldsymbol{b}')(\boldsymbol{a}+\boldsymbol{b}) \\
   &= \boldsymbol{a}'\boldsymbol{a} + \boldsymbol{a}'\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{a} + \boldsymbol{b}'\boldsymbol{b} = \boldsymbol{a}'\boldsymbol{a} + 2\boldsymbol{a}'\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{b} \\
   &= \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 + 2\boldsymbol{a}'\boldsymbol{b} \\
   &= \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 \text{ (because } \boldsymbol{a}'\boldsymbol{b} = 0\text{)}. \\
   &> \|\boldsymbol{a}\|^2 \text{ (because } \boldsymbol{b} \neq \boldsymbol{0}\text{)}.
   \end{aligned}$$
   Now note that
   $$\|\boldsymbol{y}-\boldsymbol{z}\|^2 = \|\boldsymbol{y} - \boldsymbol{P_X}\boldsymbol{y} + \boldsymbol{P_X}\boldsymbol{y} - \boldsymbol{z}\|^2 = \ldots$$

3. Suppose $\boldsymbol{X}$ is an $n \times p$ matrix. Prove that $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{P_X})$.

4. Prove that $(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y}$ is a solution for $\boldsymbol{b}$ in the Normal Equations: $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}$.

5. Suppose the Gauss-Markov model with normal errors (GMMNE) holds.

   (a) Suppose $\boldsymbol{C\beta}$ is estimable. Derive the distribution of $\boldsymbol{C}\hat{\boldsymbol{\beta}}$, the OLSE of $\boldsymbol{C\beta}$.

   (b) Now suppose $\boldsymbol{C\beta}$ is NOT estimable. Provide a fully simplified expression for $\text{Var}\left(\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y}\right)$.

   (c) Now suppose $H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$ is testable and that $\boldsymbol{C}$ has only one row and $\boldsymbol{d}$ has only one element so that they may be written as $\boldsymbol{c}'$ and d, respectively. Prove the result on slide 23 of slide set 2.