1. The easiest way to complete parts (a) through (d) is to write a few lines of SAS code as on slide 64 of slide set 8. For example,

```
proc glm;
  class source percent;
  model lconc = source percent source*percent /ss1 ss2 ss3;
  lsmeans source percent;
run;
```

Below you can see some other ways to do the computations.

```
<define functions needed for obtaining ANOVA table>
> ##projection matrix
> proj=function(X){X%*%MASS::ginv(t(X)%*%X)%*%t(X)}

> ##com: give one complete row in ANOVA table
> com=function(y,A,B){
+   ss=t(y)%*%(proj(A)-proj(B))%*%y
+   df=floor(Matrix::rankMatrix(A)-Matrix::rankMatrix(B))
+   ms=ss/df
+   f=ms/0.01351574
+   p=pf(f,df,17,lower.tail=FALSE)
+   return(round(c(ss,df,ms,f,p),4))
+ }

<data set>
> d = data.frame(pigs, lconc = log(pigs$conc), perc = factor(pigs$percent))

<generate model matrix>
> x1=matrix(rep(1,nrow(d)))
> xa=model.matrix(~0+d$source)
> xb=model.matrix(~0+d$perc)
> xab=model.matrix(~0+d$perc:d$source)
```

(a) ANOVA Table with Type 1:

```
> src1=com(d$lconc,cbind(x1,xa),x1)
> prct1=com(d$lconc,cbind(x1,xa,xb),cbind(x1,xa))
> srcprct1=com(d$lconc,cbind(x1,xa,xb,xab),cbind(x1,xa,xb))
> anova1=rbind(src1,prct1,srcprct1,error,total)
> rownames(anova1)=c("source|1","percent|1,source",
> "sourceXpercent|1,source,percent","error","corrected total")
> colnames(anova1)=c("SS","df","MS","F","p-value")
```

```
> anova1
```

	SS	df	MS	F	p-value
source|1	0.63010	2	0.31510	23.3113	< 0.0001
percent|1,source	0.31740	3	0.10580	7.8269	0.0017
source×percent|1,source,percent	0.07510	6	0.01250	0.9259	0.5011
error	0.22977	17	0.01352		
corrected total	1.25237	28			

```
<Another way>
> o=lm(lconc~source+perc+source*perc, data=d)
> anova(o)
```

Comment: The arguments A and B in function "com" are matrices such that $\mathcal{C}(\boldsymbol{B}) \subset \mathcal{C}(\boldsymbol{A})$ and calculate the sum of squares with $\boldsymbol{y}'(\boldsymbol{P}_A - \boldsymbol{P}_B)\boldsymbol{y}$.

(b) ANOVA Table with Type 2:

```
> src2=com(d$lconc,cbind(x1,xb,xa),cbind(x1,xb))
> prct2=com(d$lconc,cbind(x1,xa,xb),cbind(x1,xa))
> srcprct2=com(d$lconc,cbind(x1,xa,xb,xab),cbind(x1,xa,xb))
> anova2=rbind(src2,prct2,srcprct2,error,total)
> rownames(anova2)=c("source|1,percent","percent|1,source",
> "sourceXpercent|1,source,percent","error","corrected total")
> colnames(anova2)=c("SS","df","MS","F","p-value")
> anova2
```

	SS	df	MS	F	p-value
source|1,percent	0.76480	2	0.38240	28.2914	< 0.0001
percent|1,source	0.31740	3	0.10580	7.8269	0.0017
source×percent|1,source,percent	0.07510	6	0.01250	0.9259	0.5011
error	0.22977	17	0.01352		
corrected total	1.38707	28			

```
<Another way>
car::Anova(o, type="II")
```

(c) You can use simple R code below that makes use of the joint_tests function in the emmeans package to get the Type III sums of squares ANOVA table.

```
> d = data.frame(pigs, lconc = log(pigs$conc), perc = factor(pigs$percent))
> o = lm(lconc ~ source + perc + source:perc, data =d)
> joint_tests(emmeans(o, c("source", "perc")), test = "F")
 model term  df1 df2 F.ratio p.value
 source        2  17  30.256 <.0001
 perc          3  17   8.214 0.0013
 source:perc   6  17   0.926 0.5011
```

Comments: ANOVA Table with Type 3 obtained by using the function "Anova" in car packages does not match with the table defined in slide 60 of set 8. Thus, we can get the sums of squares, SS(source|1, percent, source:percent) and SS(percent|1, source, source:percent), by applying the approach in the slides 70∼74 of set 8.

As following the notation defined in (d), we can remove the main effect of source from the cell-means model.(see slide 72 of set 8)

$$\bar{\mu}_{1.} = \bar{\mu}_{2.} = \bar{\mu}_{3.} \Longleftrightarrow \begin{cases} \mu_{24} = \mu 11 + \mu 12 + \mu 13 + \mu 14 - \mu 21 - \mu 21 - \mu 23 \\ \mu_{34} = \mu 11 + \mu 12 + \mu 13 + \mu 14 - \mu 31 - \mu 32 - \mu 33 \end{cases}$$

From the above, we can get the reduced matrix by removing two columns corresponding to μ_{24} and μ_{34} in cell-means model matrix and replacing the rows corresponding to μ_{24} and μ_{34} with $(1,1,1,1,-1,-1,-1,0,0,0)$ and $(1,1,1,1,0,0,0,-1,-1,-1)$, respectively. The function "src.red" below R code generates the reduced matrix of source.

Similarly, we can remove the main effect of percent.

$$\bar{\mu}_{.1} = \bar{\mu}_{.2} = \bar{\mu}_{.3} = \bar{\mu}_{.4} \Longleftrightarrow \begin{cases} \mu_{32} = \mu 11 - \mu 12 + \mu 21 - \mu 22 + \mu 31 \\ \mu_{33} = \mu 11 - \mu 13 + \mu 21 - \mu 23 + \mu 31 \\ \mu_{34} = \mu 11 - \mu 14 + \mu 21 - \mu 24 + \mu 31 \end{cases}$$

From the above, we can get the reduced matrix by removing three columns corresponding to μ_{32}, μ_{33} and μ_{34} in cell-means model matrix and replacing the rows corresponding to μ_{32}, μ_{33} and μ_{34} with $(1,-1,0,0,1,-1,0,0,1)$, $(1,0,-1,0,1,0,-1,0,1)$ and $(1,0,0,-1,1,0,0,-1,1)$, respectively. The function "prct.red" below generate the reduced matrix of percent.

```
<A Model Matrix for Model with 1, percent, source*percent>
> src.red=function(x,dat){
+   new.ab=x[,c(-8,-12)]
+   for(i in 1:nrow(x)){
+     if(dat[i,1]=="soy"&dat[i,2]=="18")
+     {new.ab[i,]=c(rep(1,4),rep(-1,3),rep(0,3))}else
+     if(dat[i,1]=="skim"&dat[i,2]=="18")
+     {new.ab[i,]=c(rep(1,4),rep(0,3),rep(-1,3))}
+   }
+   return(new.ab)
+ }
> xs.red=src.red(xab,d)
> anova(lm(d$lconc~0+xs.red),lm(d$lconc~0+xab))
> src3=c(0.81788,2,0.81788/2,30.256,2.507e-06)

<A Model Matrix for Model with 1, source, source*percent>
> src3=c(0.81788,2,0.81788/2,30.256,2.507e-06)
> prct.red=function(x,dat){
+   new.ab=x[,c(-10,-11,-12)]
+   for(i in 1:nrow(x)){
+     if(dat[i,1]=="skim"&dat[i,2]=="12"){new.ab[i,]=c(1,-1,0,0,1,-1,0,0,1)}else
+     if(dat[i,1]=="skim"&dat[i,2]=="15"){new.ab[i,]=c(1,0,-1,0,1,0,-1,0,1)}else
+     if(dat[i,1]=="skim"&dat[i,2]=="18"){new.ab[i,]=c(1,0,0,-1,1,0,0,-1,1)}
+   }
+   return(new.ab)
+ }
> xp.red=prct.red(xab,d)
> anova(lm(d$lconc~0+xp.red),lm(d$lconc~0+xab))
> prct3=c(0.33304,3,0.33304/3,8.2137,0.001348)
```

ANOVA Table with Type 3:

	SS	df	MS	F	p-value
source|1,percent,source×percent	0.81788	2	0.4089400	30.2560	< 0.0001
percent|1,source,source×percent	0.33304	3	0.1110133	8.2137	0.0013
source×percent|1,source,percent	0.07510	6	0.01250	0.9259	0.5011
error	0.22977	17	0.01352		
corrected total	1.45579	28			

(d) Let μ_{ij} be a mean concentration of free plasma leucine for i source of protein and j, where $i = 1$ for fish meal, $i = 2$ for soybean meal and i = 3 dried skim milk and $j = 1, \ldots, 4$ for 9%, 12%, 15% and 18%, respectively. From the code below, each μ_{ij} can be estimated like in table.

	j=1 (9%)	j=2 (12%)	j=3 (15%)	j=4 (18%)
i=1 (fish)	3.24526	3.43011	3.43461	3.47529
i=2 (soy)	3.53845	3.67962	3.66940	3.75887
i=3 (skim)	3.56054	3.76485	3.90463	4.09101

LSMeans for source

fish : $\frac{\hat{\mu}_{11}+\hat{\mu}_{12}+\hat{\mu}_{13}+\hat{\mu}_{14}}{4} = \frac{3.24526+3.43011+3.43461+3.47529}{4} = 3.39632$
soy : $\frac{\hat{\mu}_{21}+\hat{\mu}_{22}+\hat{\mu}_{23}+\hat{\mu}_{24}}{4} = \frac{3.53845+3.67962+3.66940+3.75887}{4} = 3.66159$
skim : $\frac{\hat{\mu}_{31}+\hat{\mu}_{32}+\hat{\mu}_{33}+\hat{\mu}_{34}}{4} = \frac{3.56054+3.76485+3.90463+4.09101}{4} = 3.83026$

LSMeans for percent

9% : $\frac{\hat{\mu}_{11}+\hat{\mu}_{21}+\hat{\mu}_{31}}{3} = \frac{3.24526+3.53845+3.56054}{3} = 3.44808$
12% : $\frac{\hat{\mu}_{12}+\hat{\mu}_{22}+\hat{\mu}_{32}}{3} = \frac{3.43011+3.67962+3.76485}{3} = 3.62486$
15% : $\frac{\hat{\mu}_{13}+\hat{\mu}_{23}+\hat{\mu}_{33}}{3} = \frac{3.43461+3.66940+3.90463}{3} = 3.66955$
18% : $\frac{\hat{\mu}_{14}+\hat{\mu}_{24}+\hat{\mu}_{34}}{3} = \frac{3.47529+3.75887+4.09101}{3} = 3.77506$

(e) Since the model that percent is treated like a quantitative variable is the reduced model of the cell-means model, we can conduct the lack of fit test of the reduced model compared to the cell-means model. From the code below and error in ANOVA table,

$$F = \frac{(SSE_{Reduced} - SSE_{Full})/(df_{Reduced} - df_{Full})}{SSE_{Full}/df_{Full}} = \frac{(0.26291 - 0.22977)/(23 - 17)}{0.22977/17} = 0.4087.$$

Since the corresponding p-value is 0.8631, we can conclude that this model fit adequately relative to the cell-means model at significant level α=0.05.

```
> o = lm(lconc ~ source + perc + source:perc, data =d)
> o1=lm(lconc~source+percent+source*percent,data=d)
> anova(o1,o)
```

(f) The reduced model in (e) can be represented as $y_{ijk} = \mu + \alpha_i + \beta x_{ij} + \gamma_i x_{ij} + \epsilon_{ijk}$.
Based on this model, the estimated linear relationship for each source is following.

i. fish

$(\hat{\mu} + \hat{\alpha}_1) + (\hat{\beta} + \hat{\gamma}_1) * x_{1j} = 3.1164 + 0.0211 x_{1j}$

ii. soy

$(\hat{\mu} + \hat{\alpha}_2) + (\hat{\beta} + \hat{\gamma}_2) * x_{2j} = (3.1164 + 0.2517) + (0.0211 + 0.0006) x_{2j} = 3.3681 + 0.0217 x_{2j}$

iii. skim

$$(\hat{\mu}+\hat{\alpha}_3)+(\hat{\beta}+\hat{\gamma}_3)*x_{3j} = (3.1164-0.0672)+(0.0211+0.0369)x_{3j} = 3.0492+0.058x_{3j}$$

2. (a) Describe the distribution of these differences.

Based on the model assumptions of $e_{ij} \overset{iid}{\sim} N(0,\sigma_e^2)$, for each subject $j = 1,\cdots,20$,

$$\begin{aligned} d_j &= y_{1j} - y_{2j} \\ &= \mu_1 + u_j + e_{1j} - (\mu_2 + u_j + e_{2j}) \\ &= (\mu_1 - \mu_2) + e_{1j} - e_{2j} \end{aligned}$$

$E(d_j) = \mu_1 - \mu_2$, $Var(d_j) = Var(e_{1j}) + Var(e_{2j}) = 2\sigma_e^2$. Because a linear combination of independent normal distributions is still normal, we have $d_j \sim N(\mu_1-\mu_2, 2\sigma_e^2)$.
For any $j \neq j'$, $Cov(d_j, d_{j'}) = Cov(e_{1j}-e_{2j}, e_{1j'}-e_{2j'}) = 0$, so all d_j's are independent. Therefore $d_j \overset{iid}{\sim} N(\mu_1-\mu_2, 2\sigma_e^2)$, which is a constant mean model. We can write this as a special case of a Gauss-Markov model as follows:

$$\boldsymbol{d} = \mathbf{1}[\mu_1-\mu_2] + \boldsymbol{\epsilon}, \text{ where } \boldsymbol{d} = (d_1,\ldots,d_{20})' \text{ and } \boldsymbol{\epsilon} \sim N(\mathbf{0}, 2\sigma_e^2\boldsymbol{I}).$$

(b) Provide a formula for a test statistic (as a function of $d_1,\cdots,d_{20}$) to test $H_0: \mu_1 = \mu_2$.

Given the Gauss-Markov model above, we can find the formula for a test statistic by considering either a t test or and F test of $H_0: \boldsymbol{C\beta} = \mathbf{0}$. The general formulas for a Gauss-Markov model can be simplified in this case because the "$\boldsymbol{X}$" matrix is just $\mathbf{1}$, the "$\boldsymbol{\beta}$" vector is just the one-element vector with $\mu_1-\mu_2$ as the only element, and the "$\boldsymbol{C}$" matrix is just the 1×1 matrix with the element 1. Alternatively, can rewrite the model for differences as $d_1,\ldots,d_{20} \overset{iid}{\sim} N(\mu_d, \sigma_d^2)$, where $\mu_d = \mu_1-\mu_2, \sigma_d^2 = 2\sigma_e^2$. Now the null hypothesis is equivalent to $H_0: \mu_1-\mu_2 = \mu_d = 0$. We can now see this as a STAT 101 type of question that asks us to test whether the mean of a normal distribution is zero based on an i.i.d. sample.

Let $\bar{d}. = \dfrac{\sum_{j=1}^{20} d_j}{20}$. Then $\bar{d}. \sim N\left(\mu_d, \dfrac{\sigma_d^2}{20}\right)$, and we can build up a t statistic to test $H_0 = \mu_d = 0$ as follows:

$$\begin{aligned} t &= \frac{\bar{d}. - 0}{\sqrt{\widehat{Var}(\bar{d}.)}} \\ &= \frac{\bar{d}.}{\sqrt{\hat{\sigma}_d^2/20}} \\ &= \frac{\bar{d}.}{\sqrt{\left[\frac{1}{20-1}\sum_{j=1}^{20}(d_j-\bar{d}.)^2\right]/20}} \end{aligned}$$

Or use F test statistic $F = t^2 = \dfrac{380\,\bar{d}.^2}{\sum_{j=1}^{20}(d_j-\bar{d}.)^2}$

(c) Fully state the exact distribution of the test statistic provided in part (b).

$$t \sim t_{19}\left(\frac{\mu_d}{\sqrt{\sigma_d^2/20}}\right) \stackrel{d}{=} t_{19}\left(\frac{\mu_1-\mu_2}{\sqrt{\sigma_e^2/10}}\right)$$

$$F \sim F_{1,19}\left(\frac{5(\mu_1-\mu_2)^2}{\sigma_e^2}\right)$$

(d) Provide a formula for a 95% confidence interval for $\mu_1 - \mu_2$.
Given only the 40 scores of the subjects who received only drink one type, the model for these scores is simplfied to be a Markov model as

$$\boldsymbol{y} = \underbrace{[\boldsymbol{I}_{2\times 2} \otimes \boldsymbol{1}_{20\times 1}]}_{\boldsymbol{X}} \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + \boldsymbol{\varepsilon}$$

with $\boldsymbol{y} = [a_1, \cdots, a_{20}, b_1, \cdots, b_{20}]'$ and $\boldsymbol{\varepsilon}$ is a vector of random errors $[\varepsilon_{11}, \ldots, \varepsilon_{1,20}, \varepsilon_{21}, \ldots, \varepsilon_{2,20}]'$ where $\varepsilon_{ik} \stackrel{iid}{\sim} N(0, \sigma_u^2 + \sigma_e^2)$ for $i = 1, 2; k = 1, \cdots, 20$.
So the BLUE for $\mu_1 - \mu_2$ is $\bar{a}. - \bar{b}.$.

$$\begin{aligned}\widehat{Var}(\bar{a}. - \bar{b}.) &= \widehat{Var(\bar{a}.) + Var(\bar{b}.)} \\ &= 2 \times \frac{1}{20}\widehat{(\sigma_u^2 + \sigma_e^2)} \qquad \text{MSE for the Markov model above} \\ &= \frac{1}{10}\cdot\frac{1}{40-2}\left(\sum_{j=1}^{20}(a_j - \bar{a}.)^2 + \sum_{j=1}^{20}(b_j - \bar{b}.)^2\right)\end{aligned}$$

Therefore the 95% confidence interval for $\mu_1 - \mu_2$ is

$$(\bar{a}. - \bar{b}.) + t_{38,\,0.975}\sqrt{\frac{1}{380}\left(\sum_{j=1}^{20}(a_j - \bar{a}.)^2 + \sum_{j=1}^{20}(b_j - \bar{b}.)^2\right)}$$

with $df = n - rank(\boldsymbol{X}) = 38$

(e) Provide formulas for unbiased estimators of σ_u^2 and σ_e^2

From part (b), we have $\hat{\sigma}_d^2 = 2\hat{\sigma}_e^2 = \frac{1}{20-1}\sum_{j=1}^{20}(d_j - \bar{d}.)^2$.
From part (d) we have $\widehat{\sigma_u^2 + \sigma_e^2} = \frac{1}{40-2}\left(\sum_{j=1}^{20}(a_j - \bar{a}.)^2 + \sum_{j=1}^{20}(b_j - \bar{b}.)^2\right)$.
By solving the equations above, we can obtain

$$\begin{cases}\hat{\sigma}_e^2 = \dfrac{\sum_{j=1}^{20}(d_j - \bar{d}.)^2}{38} \\ \hat{\sigma}_u^2 = \dfrac{\left(\sum_{j=1}^{20}(a_j - \bar{a}.)^2 + \sum_{j=1}^{20}(b_j - \bar{b}.)^2\right)}{38} - \dfrac{\sum_{j=1}^{20}(d_j - \bar{d}.)^2}{38}\end{cases}$$

(f) Provide a simplified expression for the best linear unbiased estimator of $\mu_1 - \mu_2$.

Both $\bar{d}.$ and $(\bar{a}. - \bar{b}.)$ are independent unbiased estimators of $\mu_1 - \mu_2$. Thus, the BLUE of $\mu_1 - \mu_2$ is the weighted average of $\bar{d}.$ and $(\bar{a}. - \bar{b}.)$ with weights proportional to the inverse of the variances.

$$\begin{aligned}
\widehat{\mu_1 - \mu_2} &= \frac{Var^{-1}(\bar{d}.)}{Var^{-1}(\bar{d}.) + Var^{-1}(\bar{a}. - \bar{b}.)} \cdot \bar{d}. + \frac{Var^{-1}(\bar{a}. - \bar{b}.)}{Var^{-1}(\bar{d}.) + Var^{-1}(\bar{a}. - \bar{b}.)} \cdot (\bar{a}. - \bar{b}.) \\
&= \frac{\sigma_u^2 + \sigma_e^2}{\sigma_u^2 + 2\sigma_e^2} \cdot \bar{d}. + \frac{\sigma_e^2}{\sigma_u^2 + 2\sigma_e^2} \cdot (\bar{a}. - \bar{b}.)
\end{aligned}$$

3. Suppose the responses in problem 2 were sorted first by subject and then by drink, the response vector $\boldsymbol{y} = [y_{11}, y_{21}, \cdots, y_{1,20}, y_{2,20}, y_{1,21}, \cdots, y_{1,40}, y_{2,41}, \cdots, y_{2,60}]'$.
In model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}$,

$$\boldsymbol{X} = \left[\begin{array}{cc}
1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \hline
1 & \\ 1 & \\ \vdots & \\ 1 & \\ \hline
 & 1 \\ & 1 \\ & \vdots \\ & 1
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right\} \text{40 rows} \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} \text{20 rows} \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} \text{20 rows}
\end{array}
\quad \text{and } \boldsymbol{Z} = \left[\begin{array}{ccccc|ccc|ccc}
1 & & & & & & & & & & \\
1 & & & & & & & & & & \\
 & 1 & & & & & & & & & \\
 & 1 & & & & & & & & & \\
 & & \ddots & & & & & & & & \\
 & & & & 1 & & & & & & \\
 & & & & 1 & & & & & & \\
 & & & & & 1 & & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & & 1 & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{array}\right]$$

the Kronecker product notation for $\boldsymbol{X}$ and $\boldsymbol{Z}$ are

$$\boldsymbol{X}_{80\times 2} = \begin{bmatrix} \boldsymbol{1}_{20\times 1} \otimes \boldsymbol{I}_{2\times 2} \\ \boldsymbol{I}_{2\times 2} \otimes \boldsymbol{1}_{20\times 1} \end{bmatrix}$$

$$\boldsymbol{Z}_{80\times 60} = \begin{bmatrix} \boldsymbol{I}_{20\times 20} \otimes \boldsymbol{1}_{2\times 1} & \boldsymbol{O}_{40\times 40} \\ \boldsymbol{O}_{40\times 20} & \boldsymbol{I}_{40\times 40} \end{bmatrix}$$

4. (a)

$$
\begin{aligned}
EMS_{xu(trt)} &= \frac{1}{df_{xu(trt)}} \operatorname{E}\left(SS_{xu(trt)}\right) \\
&= \frac{1}{tn-t} \operatorname{E}\left(m \sum_{i=1}^{t} \sum_{j=1}^{n} (y_{ij\boldsymbol{.}} - \bar{y}_{i\boldsymbol{..}})^2 \right) \\
&= \frac{1}{tn-t} \operatorname{E}\left(m \sum_{i=1}^{t} \sum_{j=1}^{n} ([\mu + \tau_i + u_{ij} + \bar{e}_{ij\boldsymbol{.}}] - [\mu + \tau_i + \bar{u_{i\boldsymbol{.}}} + \bar{e}_{i\boldsymbol{..}}])^2 \right) \\
&= \frac{m}{tn-t} \sum_{i=1}^{t} \sum_{j=1}^{n} \operatorname{E}\left\{ (u_{ij} - \bar{u}_{i\boldsymbol{.}}) + (\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}}) \right\}^2 \\
&= \frac{m}{tn-t} \sum_{i=1}^{t} \sum_{j=1}^{n} \left\{ \operatorname{E}(u_{ij} - \bar{u}_{i\boldsymbol{.}})^2 + \operatorname{E}(\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}})^2 \right\} && \text{see the comment} \\
&= \frac{m}{tn-t} \sum_{i=1}^{t} \left[\operatorname{E}\left\{ \sum_{j=1}^{n} (u_{ij} - \bar{u}_{i\boldsymbol{.}})^2 \right\} + \operatorname{E}\left\{ \sum_{j=1}^{n} (\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}})^2 \right\} \right] \\
&= \frac{m}{tn-t} \sum_{i=1}^{t} \left\{ (n-1)\sigma_u^2 + (n-1)\frac{\sigma_e^2}{m} \right\} && \text{since } u_{ij} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma_u^2\right) \text{ and } \bar{e}_{ij\boldsymbol{.}} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \frac{\sigma_e^2}{m}\right) \\
&= \frac{m}{tn-t} \left\{ t(n-1)\sigma_u^2 + t(n-1)\frac{\sigma_e^2}{m} \right\} \\
&= m\sigma_u^2 + \sigma_e^2.
\end{aligned}
$$

Comment:

$$
\begin{aligned}
\operatorname{E}\left\{ (u_{ij} - \bar{u}_{i\boldsymbol{.}}) + (\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}}) \right\}^2 &= \operatorname{Var}\Big((u_{ij} - \bar{u}_{i\boldsymbol{.}}) + (\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}}) \Big) && \text{since } \operatorname{E}(u_{ij} - \bar{u}_{i\boldsymbol{.}}) = \operatorname{E}(\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}}) = 0 \\
&= \operatorname{Var}(u_{ij} - \bar{u}_{i\boldsymbol{.}}) + \operatorname{Var}(\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}}) && \text{since by assumption in slide 2 of set 12} \\
&= \operatorname{E}(u_{ij} - \bar{u}_{i\boldsymbol{.}})^2 + \operatorname{E}(\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}})^2
\end{aligned}
$$

(b) We can show this in a general case for t, n, m first. From slide 6, the sum of squares can be written as $\boldsymbol{y}'(\boldsymbol{P}_3 - \boldsymbol{P}_2)\boldsymbol{y}$, where

$$
\begin{aligned}
\boldsymbol{P}_2 &= [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]\Big([\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]'[\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}] \Big)^{-} [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]' \\
&= [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}] \begin{bmatrix} tnm & nm\mathbf{1}'_{t\times 1} \\ nm\mathbf{1}_{t\times 1} & nm\boldsymbol{I}_{t\times t} \end{bmatrix}^{-} [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]' \\
&= [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}] \begin{bmatrix} 0 & \mathbf{0}'_{t\times 1} \\ \mathbf{0}_{t\times 1} & \frac{1}{nm}\boldsymbol{I}_{t\times t} \end{bmatrix} [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]' \\
&= \frac{1}{nm}\boldsymbol{I}_{t\times t} \otimes \mathbf{1}\mathbf{1}'_{nm\times nm}
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{P}_3 &= [\boldsymbol{I}_{tn\times tn}\otimes \mathbf{1}_{m\times 1}]\Big([\boldsymbol{I}_{tn\times tn}\otimes \mathbf{1}_{m\times 1}]'[\boldsymbol{I}_{tn\times tn}\otimes \mathbf{1}_{m\times 1}]\Big)^{-1}[\boldsymbol{I}_{tn\times tn}\otimes \mathbf{1}_{m\times 1}]' \\
&= [\boldsymbol{I}_{tn\times tn}\otimes \mathbf{1}_{m\times 1}]\Big(m\boldsymbol{I}_{tn\times tn}\Big)^{-1}[\boldsymbol{I}_{tn\times tn}\otimes \mathbf{1}_{m\times 1}]' \\
&= \tfrac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m}.
\end{aligned}
$$

Let

$$
\boldsymbol{A} = \frac{\boldsymbol{P}_3-\boldsymbol{P}_2}{t(n-1)} = \frac{\frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m} - \frac{1}{nm}\boldsymbol{I}_{t\times t}\otimes \mathbf{11}'_{nm\times nm}}{t(n-1)}.
$$

Then, by linearity of trace,

$$
\begin{aligned}
\operatorname{tr}(\boldsymbol{A\Sigma}) &= \operatorname{tr}\left(\left[\frac{\frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m} - \frac{1}{nm}\boldsymbol{I}_{t\times t}\otimes \mathbf{11}'_{nm\times nm}}{t(n-1)}\right]\left[\sigma_u^2\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m}+\sigma_e^2\boldsymbol{I}_{tnm\times tnm}\right]\right) \\
&= \frac{1}{t(n-1)nm}\operatorname{tr}\left(\left[n\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m} - \boldsymbol{I}_{t\times t}\otimes \mathbf{11}'_{nm\times nm}\right]\left[\sigma_u^2\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m}+\sigma_e^2\boldsymbol{I}_{tnm\times tnm}\right]\right) \\
&= \frac{1}{t(n-1)nm}\operatorname{tr}\Big[nm\sigma_u^2\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m} + n\sigma_e^2\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m} \\
&\qquad - \sigma_u^2(\boldsymbol{I}_{t\times t}\otimes \mathbf{11}'_{nm\times nm})(\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m}) - \sigma_e^2\boldsymbol{I}_{t\times t}\otimes \mathbf{11}'_{nm\times nm}\Big] \\
&= \frac{1}{t(n-1)nm}\Big[nm\sigma_u^2\operatorname{tr}(\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m}) + n\sigma_e^2\operatorname{tr}(\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m}) \\
&\qquad - m\sigma_u^2\operatorname{tr}(\boldsymbol{I}_{t\times t}\otimes \mathbf{11}'_{nm\times nm}) - \sigma_e^2\operatorname{tr}(\boldsymbol{I}_{t\times t}\otimes \mathbf{11}'_{nm\times nm})\Big] \\
&= \frac{1}{t(n-1)nm}(tnm)\Big(nm\sigma_u^2+n\sigma_e^2-m\sigma_u^2-\sigma_e^2\Big) \quad \text{since } \boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m} \text{ and } \boldsymbol{I}_{t\times t}\otimes \mathbf{11}'_{nm\times nm} \text{ have 1's on the diagonal} \\
&= \frac{1}{n-1}\Big(m\sigma_u^2(n-1)+\sigma_e^2(n-1)\Big) \\
&= m\sigma_u^2+\sigma_e^2
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{E}(\boldsymbol{y})'\boldsymbol{A}\operatorname{E}(\boldsymbol{y}) &= \operatorname{E}(\boldsymbol{y})'\left(\frac{\frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m} - \frac{1}{nm}\boldsymbol{I}_{t\times t}\otimes \mathbf{11}'_{nm\times nm}}{t(n-1)}\right)\operatorname{E}(\boldsymbol{y}) \\
&= \frac{1}{tnm(n-1)}\left(n\operatorname{E}(\boldsymbol{y})'\boldsymbol{I}_{tn\times tn}\otimes \mathbf{11}'_{m\times m} - \operatorname{E}(\boldsymbol{y})'\boldsymbol{I}_{t\times t}\otimes \mathbf{11}'_{nm\times nm}\right)\operatorname{E}(\boldsymbol{y}) \\
&= \frac{1}{tnm(n-1)}\left(n(m\operatorname{E}(\boldsymbol{y})') - nm\operatorname{E}(\boldsymbol{y})'\right)\operatorname{E}(\boldsymbol{y}) \\
&= \frac{1}{tnm(n-1)}\mathbf{0}'\operatorname{E}(\boldsymbol{y}) \\
&= 0.
\end{aligned}
$$

Now,

$$
\begin{aligned}
EMS_{ou(xu,trt)} &= \operatorname{E}\left(\boldsymbol{y}'\left(\frac{\boldsymbol{P}_3-\boldsymbol{P}_2}{t(n-1)}\right)\boldsymbol{y}\right) \\
&= \operatorname{E}(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}) \\
&= \operatorname{tr}(\boldsymbol{A\Sigma}) + \operatorname{E}(\boldsymbol{y})'\boldsymbol{A}\operatorname{E}(\boldsymbol{y}) \qquad \text{by slide 19 of set 12} \\
&= (m\sigma_u^2+\sigma_e^2)+0 \\
&= m\sigma_u^2+\sigma_e^2,
\end{aligned}
$$

The result also holds for the special case of $t=2, n=2, m=2$, which is the same result as in part (a).